Math 542 Comprehensive Examination May 17, 2017

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.
Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.
Justify all your answers. Good Luck!

1. Suppose that the Taylor series of an entire function f converges to f uniformly in $\mathbb{C}$. Prove that either f is a non-zero constant or f has a zero.

2. Let $D \subseteq \mathbb{C}$ be the open unit disk domain and $f : D \to \mathbb{C}$ be a non-constant function. Suppose that $|f(z)| = 1$ for every $z \in \partial D$. Prove that f has at least one zero in D.

3. **Use residues** to evaluate the principal value of the following integral $\int_{-\infty}^{\infty} \frac{x \cos x}{x^2-5x+6} dx$.
Justify each step.

4. Let f be an entire function such that $|f(z)| \leq (1+|z|)^{\frac{2}{3}} e^{\mathrm{Re}\, z}$ for large z. Prove that there is $c \in \mathbb{C}$ such that $f(z) = ce^z$ for every $z \in \mathbb{C}$.

5. Recall that $\binom{n}{k}$ denotes the binomial coefficient, where $n \in N$ and $k \leq n$ is a non-negative integer.
(a) Prove: $\binom{n}{k} = \frac{1}{2\pi i} \int_\gamma \frac{(1+z)^n}{z^{k+1}} dz$, where γ is a simple closed curve enclosing the origin.
(b) Prove: $\binom{2n}{n} \leq 4^n$.
(c) Use parts (a) and (b) to find the sum of the following infinite series: $\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{7^n}$.
Hint for part (c): Show that

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{7^n} = \frac{7}{2\pi i} \int_{|z|=1} \frac{dz}{5z - z^2 - 1}.$$

6. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Let $\mathcal{F}$ be the family of entire functions defined by

$$\mathcal{F} = \{g_c : C \to C : g_c(z) = f(cz), c \in C\}.$$

Fix $0 \leq r_1 < r_2 \leq +\infty$. Suppose that every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact set of the annular domain

$$\mathrm{ann}(0; r_1, r_2) = \{z \in C : r_1 < |z| < r_2\}.$$

either to a function $g : \mathbb{C} \to \mathbb{C}$ or to a function identically equal to ∞. Prove that f is a polynomial.