Math 542 Comprehensive Examination May 2019

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems. Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3, and 4. Each problem is worth 10 points. Justify all your answers. Good Luck!

Notation:
We denote the set of complex numbers by $\mathbb{C}$. We write $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ for the unit disk in $\mathbb{C}$.

1. Let Ω be a convex domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for every $z \in \Omega$. Prove that there exists an analytic function $h : \Omega \to \mathbb{C}$ such that
$$f(z) = e^{h(z)} \quad \text{for all } z \in \Omega.$$

2. Does there exist an analytic function $f : \{z \in \mathbb{C} : 0 < |z| < 1\} \to \mathbb{C}$ satisfying
$$\lim_{z\to 0} \frac{f(z)^2}{z^3} = 1\ ?$$
Justify your answer.

3. Does there exist an analytic function f of $\mathbb{D}$ into $\mathbb{D}$ such that $f'(1/2) = 2$? If yes, give one example of such a function f. If not, prove that there is no such function f.

4. Let a be a real number with $0 < a < 1$. Evaluate the integral
$$\int_0^\infty \frac{x^{-a}}{1+x}\,dx.$$
Show all the estimates required by your argument.

5. Suppose that (a_n) and (b_n) are disjoint sequences of complex numbers such that $\sum_{n=1}^{\infty} |a_n - b_n| < \infty$ and $\lim_{n\to\infty} |a_n| = \infty$. Show that $\prod_{n=1}^{\infty} \frac{z-a_n}{z-b_n}$ defines a meromorphic function on the complex plane.

6. Give an explicit example of a function f meromorphic on $\mathbb{C}$ such that f has a simple pole with residue n at each integer $n \geq 1$ and no other poles. Prove that your answer is correct.