Math 542 Comprehensive Examination May 21, 2015

Do five out of six problems. Each problem is worth 20 points. Clearly indicate which problems you are doing. You must justify all claims to receive credit.

I. (i) Find a 1-1 analytic mapping f of $\{z : |z| < 1\}$ onto $\mathbb{C} \setminus \{x : x \geq 0\}$ such that $f(0) = -1$.

(ii) Determine with proof all analytic maps f of $\{z : |z| < 1\}$ into $\mathbb{C} \setminus \{x : x \geq 0\}$ such that $f(0) = -1$ and $f(\frac{1}{2}) = -\frac{1}{9}$.

II. Evaluate

$$\int_0^\infty \frac{\cos x}{(4+x^2)^2}\, dx$$

Justify all estimates.

III. Let f be an analytic mapping defined on an open set containing $\{z : |z| \leq 1\}$. Show that the quantity

$$\int_{\{z=x+iy:|x+iy|\leq 1\}} f(x+iy)\, dx\, dy$$

depends only on $f(0)$.

IV. Does

$$g(z) = \frac{\sin z}{z^{100}}$$

have an antiderivative in the domain $\{z : 1 < |z| < 2\}$? Justify your answer.

V. Let f be a meromorphic function in $\mathbb{C}$ and let γ be a simple closed curve in $\mathbb{C}$ on which f does not take on the values 0 or ∞. Prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = N - P,$$

where N is the number of zeros of f inside γ counting multiplicity and P is the number of poles of f inside γ counting multiplicity.

VI. (i) Define the Weierstrass primary factor $E_p(z)$ of genus p for $p = 0, 1, 2, \ldots$

(ii) Show that

$$\prod_{n=1}^{\infty} E_{n-1}\left(\frac{1}{nz}\right)$$

defines an analytic function on $\{z : \operatorname{Re} z > 0\}$ which vanishes precisely on

$$\{\frac{1}{n} : n = 1, 2, 3 \ldots\}.$$