Math 542 Comprehensive Examination

May 2014

Solve all eight problems. All problems have the same weight. **Justify all claims.**

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$ in the complex plane and $\operatorname{Re} z$, $\operatorname{Im} z$ denote the real and respectively imaginary part of $z \in \mathbb{C}$.

1. Is there an analytic unction f in the unit disc $\mathbb{D}$ such that
$$f\left(\frac{1}{n}\right)^2 = \frac{1}{n}, \qquad n = 2, 3, 4, \ldots ?$$

2. If $f : \mathbb{D} \to \mathbb{D}$ is analytic and has order 2 at $z = 0$, what is the sharp upper bound for $|f(\frac{1}{2})|$?

3. Calculate
$$I = \int_0^\pi \frac{d\theta}{3 + \sin(2\theta)}.$$

4. Find the number of solutions in $\mathbb{D}$ of
$$z^4 - 7z^3 - 2z^2 + z - 3 = 0.$$

5. Can the function
$$f(z) = z \sin\left(\frac{1}{z}\right)$$
be uniformly approximated by polynomials on compact subsets of $\mathbb{C} \setminus \{0\}$? Prove or give a counterexample.

6. Is there a positive harmonic function u in the first quadrant $I = \{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$, such that u is continuous up to the boundary of I and $u(x, 0) = x^2$ for all $x > 0$ and $u(0, iy) = y^2$ for all $y > 0$?

7. Which of the following two families of analytic functions in $\mathbb{D}$ is normal:
$$\mathfrak{F}_1 = \left\{ f \in A(\mathbb{D}) : \iint_{\mathbb{D}} |f(x + iy)|^2 dx\, dy \ \le \ 1 \right\},$$
$$\mathfrak{F}_2 = \left\{ f \in A(\mathbb{D}) : \iint_{\mathbb{D}} |f(x + iy)|^2 dx\, dy \ < \ \infty \right\} ?$$

8. Prove the identity
$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z - n} + \frac{1}{z + n} \right).$$