# Math 542 Comprehensive Examination

May 2013

**Each problem is worth 10 points. Justify all the claims that you make.**

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$, and $\mathbb{T}$ the unit circle $\{z \in \mathbb{C}\colon |z| = 1\}$.

1. Suppose that $f$ is an analytic function in a domain $D$ such that $f'(z_0) = 0$ for some $z_0 \in D$. Prove that $f$ is not one-to-one in any neighborhood of $z_0$.

2. Let $f\colon \mathbb{D} \setminus \{0\} \to \mathbb{C}$ be an analytic function. Is it true that $f$ has a removable singularity at 0 if and only if the function $e^f$ has a removable singularity at 0?

3. Find the radius of convergence of the power series
$$\sum_{k=1}^{\infty} k^k z^{k^2}.$$

4. Suppose that $f$ and $g$ are entire functions and there exists $R > 0$ such that $|f(z)| \le |g(z)|$ for all $z$ with $|z| > R$. Prove that $f/g$ is a rational function.

5. Evaluate
$$\int_0^{\pi/2} \frac{d\theta}{3 - \cos^2\theta}.$$

6. Find a conformal map from the complement of the cross
$$\mathbb{C} \setminus (\{z = x + i0\colon\ -1 \le x \le 1\} \cup \{z = 0 + iy\colon\ -1 \le y \le 1\})$$
onto the complement of the closed unit disc $\mathbb{C} \setminus \overline{\mathbb{D}}$.

7. Let $u\colon \mathbb{D} \setminus \{0\} \to \mathbb{R}$ be a harmonic function. Prove that there exists a unique real number $\alpha$ such that $u(z) - \alpha \ln |z|$ is the real part of an analytic function $f\colon \mathbb{D} \setminus \{0\} \to \mathbb{C}$.

8. Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function that is analytic in $\mathbb{D}$. Assume that $f$ maps the unit circle $\mathbb{T}$ into $\mathbb{T} \setminus \{-1\}$. Prove that $f$ is a constant.