Math 542 Comprehensive Examination

May 2012

Solve **EIGHT** of the following **NINE** problems. Indicate the problems you want graded.
For $z \in \mathbb{C}$, let $\Re z$ and $\Im z$ denote the real and imaginary parts of z, respectively.

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$ and $\mathbb{T}$ the unit circle $\{z \in \mathbb{C} \colon |z| = 1\}$ in the complex plane $\mathbb{C}$.

1. Is there an analytic function f in the punctured disc $\mathbb{D} \setminus \{0\}$ such that f' has a simple pole at 0? Justify your claim.

2. Use the Residue Theorem to calculate the integral
$$\int_0^\infty \frac{x^2\sqrt{x}}{(x^2+1)^2}\,dx.$$

3. Let f be an entire function such that there exist positive constants $C, R > 0$ with
$$|\Re f(z)| \leqslant C|\Im f(z)|$$
for all z, $|z| > R$. Prove that f is a constant.

4. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. Assuming that there exist $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$, such that $f(z_1) = z_1$ and $f(z_2) = z_2$, prove that $f(z) = z$ for all $z \in \mathbb{D}$.

5. What is the number of solutions of
$$z^4 - 4z^3 + 6z - 3 = 0$$
in $\{z \in \mathbb{C} \colon |z| < 2,\ \Im z > 0\}$? Justify your claim.

6. Suppose that (f_n) is a sequence of analytic functions on a domain D. Prove that L^2-convergence of (f_n) on compact subsets of D implies normal convergence (i.e., uniform convergence on compacts) on D together with all derivatives.

7. Prove or disprove the following statement:
There exists a sequence of polynomials $\big(p_n(z)\big)_n$ such that $p_n(z)$ converges uniformly on the unit circle $\mathbb{T}$ to the function $f(z) = \bar{z}^2$.

8. Let u be a harmonic function in $\mathbb{R}^2$ such that

$$u(x,y) + y^2 \geqslant x^2, \quad \forall\,(x,y) \in \mathbb{R}^2.$$

Prove that there exists a non-negative constant C such that

$$u(x,y) + y^2 = x^2 + C, \quad \forall\,(x,y) \in \mathbb{R}^2.$$

9. Construct an entire function $f(z)$ that has simple zeros at $\sqrt[4]{n}$, $n = 0, 1, 2, 3, \ldots$ and at $\pm im^2$, $m = \pm 1, \pm 2, \ldots$, and no other zeros. If $g(z)$ is another function with the same properties, what is the relation between $f(z)$ and $g(z)$?