Math 542 Comprehensive Examination

May 18, 2011

Solve **any eight** of the following **nine** problems. Each problem is worth 10 points. $\mathbb{D}$ denotes the open unit disc in the complex plane $\mathbb{C}$.

1. Is there an analytic function
$$f(z) = u(x,y) + iv(x,y), \quad z = x + iy,$$
in a neighborhood of $z = 0$ such that
$$u(x,y) = x^3 + xy^2?$$
Justify your claim.

2. Prove that every function u, harmonic on a simply connected domain Ω, can be represented in the form $u(x,y) = \log|f(z)|$ with $z = x + iy$, where $f(z)$ is analytic and non-vanishing on Ω. Is the result true when Ω is an annulus ?

3. Find the number of solutions to the equation
$$\log z + 8z^2 = 10$$
in $\operatorname{Re} z > 0$, where $\log z$ denotes the principal branch.

4. Prove that the function
$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = 1 + z^2 + z^4 + z^8 + \cdots, \quad z \in \mathbb{D},$$
does not extend analytically to any open set strictly larger than the (open) disk $\mathbb{D}$.

5. Compute
$$\operatorname{Re} \int_\gamma \frac{\sqrt{z}}{z+1} dz,$$
where γ is a quarter-circle $\{z \colon |z| = 1, \ \operatorname{Re} z \geqslant 0, \operatorname{Im} z \geqslant 0\}$ oriented from 1 to i, and $\sqrt{z}$ denotes the principal branch.

6. Let $f : \mathbb{D} \to \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ be analytic. Show that
$$\frac{1-|z|}{1+|z|}\,|f(0)| \leqslant |f(z)| \leqslant \frac{1+|z|}{1-|z|}\,|f(0)|, \quad \forall z \in \mathbb{D},$$
and
$$|f'(0)| \leqslant 2 \operatorname{Re} f(0).$$

7. Find a conformal map from $\{z \in \mathbb{C} \colon |z| > 1, \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ to the upper half-plane.

8. Let U be a domain in $\mathbb{C}$ and $z_0 \in U$. Let $\mathcal{F}$ be the family of analytic functions f in U such that $f(z_0) = -1$ and $f(U) \cap \mathbb{Q}_{\geqslant 0} = \emptyset$, where $\mathbb{Q}_{\geqslant 0}$ denotes the set of non-negative rational numbers. Is $\mathcal{F}$ a normal family? Justify your claim.

9. Let u be a continuous real-valued function in the closure of the unit disc $\mathbb{D}$ that is harmonic in $\mathbb{D}$. Assume that the boundary values of u are given by

$$u(e^{it}) = 5 - 4\cos t.$$

Furthermore, let v be a harmonic conjugate of u in $\mathbb{D}$ such that $v(0) = 1$. Find $u(1/2)$ and $v(1/2)$.