Math 542 **Comprehensive Examination** **May 2009**

Do any four of the following five problems. Only four problems will be graded. Indicate clearly which problem is not to be graded. Each problem is worth 25 points. Justify all your answers. Good luck!

Notation:
C denotes the set of all complex numbers.

1. Evaluate
$$\int_0^\infty \frac{x \sin x \, dx}{1+x^2}.$$

2. Find a conformal map from the strip $\{z = x + iy : 0 < y < 1\}$ onto the half-strip $\{z = x + iy : x > 0,\ 0 < y < 1\}$. (Hint. Composition of several elementary conformal maps.)

3. (a) Prove that the function $f(z) = 1/z$ does not have an anti-derivative in $\mathbf{C} \setminus \{0\}$.

   (b) Find all integers $n = 0, \pm 1, \pm 2, \ldots$ such that the function $g(z) = z^n e^{1/z}$ has an anti-derivative in $\mathbf{C} \setminus \{0\}$.

4. Let f be an analytic function with a zero of order 2 at $z_0 \in \mathbf{C}$. Prove there exist $\epsilon > 0$ and $\delta > 0$ such that for every w in the set $\{w \in \mathbf{C} : 0 < |w| < \epsilon\}$ the equation $f(z) = w$ has exactly 2 distinct roots in the set $\{z \in \mathbf{C} : 0 < |z - z_0| < \delta\}$. (Hint. Use the Rouché theorem.)

5. Let $\mathcal{H}$ denote the set of all **nonnegative** functions u continuous in the closed disk $|z| \leq 3$, harmonic in the open disk $|z| < 3$, with $u(0) = 1$. Find
$$\inf_{u \in \mathcal{H}} u(1) \qquad \text{and} \qquad \sup_{u \in \mathcal{H}} u(1).$$
(Hint. Use the Poisson integral and/or Harnack inequality.)