

Do any four of the following five problems. Only four problems will be graded. Indicate clearly which problem is not to be graded. Each problem is worth 25 points. Justify all your answers. Good luck!

Notation:

\mathbf{C} denotes the set of all complex numbers.

$A(D)$ denotes the set of all analytic functions on an open set $D \subset \mathbf{C}$ continuous up to ∂D ; here ∂D stands for the boundary of D .

1. Is there a function f analytic in $\{0 < |z| < 1\}$ satisfying

$$\lim_{z \rightarrow 0} [z^{-3} f(z)^2] = 1 \quad ?$$

Justify your answer.

2. Put $A_r = \{z \in \mathbf{C} : 1 < |z| < r\}$. Does there exist a conformal (analytic one-to-one onto) map $f : A_2 \rightarrow A_4$? Justify your answer. (You can use without proof that f extends to a homeomorphism of the closed domains.) (Hint: consider $\ln|f|$.)

3. Evaluate

$$\int_0^{\infty} \frac{dx}{x^{1/5}(x+1)}$$

by contour integration.

4. Let $D \subset \mathbf{C}$ be the unit disk with center at 0. Find all functions $f \in A(D)$ such that $\operatorname{Re} f(z) = (\operatorname{Re} z)^3$ for $|z| = 1$.
5. Let $D \subset \mathbf{C}$ be the unit disk with center at 0. Let $\mathcal{F} = \{f \in A(D) : \int_0^{2\pi} |f(e^{it})| dt \leq 1\}$.
- (a) Prove that every sequence $f_n \in \mathcal{F}$ has a subsequence converging on compact subsets of D .
- (b) Does the limit of the subsequence in (a) have to be in \mathcal{F} ? Justify your answer.