Math 542 **Comprehensive Examination** **May 18, 2007**

Solve any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. **Show your work.**

$D = \{z \in \mathbb{C} \ : \ |z| < 1\}$
$ann\,(a; R_1, R_2) = \{z \in \mathbb{C} \ : \ R_1 < |z - a| < R_2\}$
Notation: *$H(G)$ denotes the set of all analytic functions on an open set $G \subset \mathbb{C}$*
A domain is a non-empty open and connected set

1. By Liouville's theorem, a bounded entire function f is constant. Prove this theorem by calculating the integral

$$\int_{|z|=R} \frac{f(z)\,dz}{(z-a)(z-b)} \quad (|a| < R, |b| < R, a \neq b)$$

and taking its limit as $R \to \infty$.

2. Let f be a polynomial of degree 2007 such that $|f(z)| \leq 1$ if $|z| \leq 1$. Prove that $|f(z)| \leq |z|^{2007}$ if $|z| \geq 1$.

3. Let f be a polynomial of degree 2007,

$$f(z) = \sum_{n=0}^{2007} a_n z^n, \ z \in \mathbb{C}.$$

If f has exactly 1966 zeros in the unit disk D (counted according to multiplicity), prove that

$$\min_{|z|=1} |f(z)| \leq |a_0| + |a_1| + \ldots + |a_{1966}|.$$

4. Evaluate the following integral by using the Cauchy residue theorem:

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx.$$

Justify each step and all estimates used.

5. Let G be a domain in $\mathbb{C}$. Suppose that to every $a \in G$ there correspond real positive numbers $r(a)$ and $M(a)$. Define the family $\mathcal{F}$ of functions $f : G \to \mathbb{C}$ by the following condition: $f \in \mathcal{F}$ if, and only if, $f \in H(G)$ and for all $a \in G$,

$$\iint\limits_{B(a,r(a))\cap G} |f(x+iy)|\, dxdy \leq M(a).$$

Is $\mathcal{F}$ a normal family? Justify your answer completely.