Math 542 **Comprehensive Examination** **May 16, 2006**

Do any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. **Show your work.**

$B(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}$

Notation: $H(G)$ *denotes the set of all analytic functions on an open set* $G \subset \mathbb{C}$

1. Let f be meromorphic in G with a finite number of poles in G and analytic in a neighborhood of ∂G, where $G = B(0,1) \cup B(2,1/2)$. Prove that if $\operatorname{Im} f(z) \neq 0$ for every $z \in \partial G$, then the number of poles is the same as the number of zeros (counting multiplicities) of f in G.

2. Let f be a nonconstant entire function. Suppose that there is a sequence of polynomials $\{P_n(z)\}_{n=1,2,\ldots}$ such that
(i) $P_n(z)$ converges uniformly to $f(z)$ on every bounded set in $\mathbb{C}$;
(ii) For every n, each zero of $P_n(z)$ is real.
Prove that all zeros of $f(z)$ are real.

3. Let $f(z)$ be meromorphic on $\mathbb{C}$, and suppose $\lim_{z\to\infty} |f(z)| = \infty$. Prove that f is a rational function.

4. Give an explicit series representation for a function f that is meromorphic on $\mathbb{C}$ with a simple pole at $z = -\sqrt{n}$ with $\operatorname{Res}_{z=-\sqrt{n}} f(z) = \sqrt{n}$, $n = 1, 2, 3, ..,$ and having no other poles. Justify your answer.

5. Let $\Omega = \{x + iy \mid x > 0 \text{ and } y > 0\}$. Let $\mathcal{F}$ be the collection of all analytic mappings of Ω into $B(0,1)$.
1) Find $\sup_{f\in\mathcal{F}} |f'(1+i)|$ and justify your answer.
2) Does there exist $f \in \mathcal{F}$ such that $f(1+i) = 0$ and $f(2+2i) = 4i/5$? Justify your answer.