Math 542 Comprehensive Examination January 18, 2018

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.
Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.
Justify all your answers. Good Luck!

1. Find all entire functions f with bounded L^1-norm in $\mathbb{C}$, i.e., satisfying

$$\iint_{\mathbb{C}} |f(x,y)|\, dxdy < +\infty.$$

Justify your answer.

2. Let the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converge in the unit disk D. Suppose that f is analytic on ∂D except for simple poles. Prove that the sequence $(a_n)_{n=1}^{\infty}$ is bounded.

3. Find all the quadrants or their boundaries containing all roots of the following equation: $z^4 + z^3 + 4z^2 + 2z + 3 = 0$ and determine the number of roots in each quadrant or its boundary. **Justify your answer completely.**

4. **Use analytic functions** to find a solution for the bounded steady temperatures in a plate having the form of the first quadrant $x \geq 0,\ y \geq 0$ if its faces are perfectly insulated and its edges have temperatures $T(x,0) = 0$ and $T(0,y) = 1$; that is, solve the following Dirichlet's problem: $\nabla^2 T = 0, x > 0, y > 0, T(x,0) = 0$ and $T(0,y) = 1; T(x,y)$ is bounded. Find isotherms (i.e., level curves of $T(x,y)$) and draw some of them (take $c = 1, 0$ and -1). **Justify your answer.**

5. Let f be a non-identically equal to zero continuous function in $\mathbb{C}$ and $(p_n)_{n=1}^{\infty}$ be a sequence of polynomials converging to f uniformly on every open disk in $\mathbb{C}$. Suppose that every polynomial p_n has only real roots. Prove that f has only real roots.

6. Let $K = \left\{z \in \mathbb{C} : |z| \leq 1 \text{ and } |z-1| \geq \frac{2}{3}\right\}$ and $G = \operatorname{int}(K)$.
(a) Is it true that polynomials are dense in $H(G)$ (w.r.t. standard uniform convergence on compact subsets of G)? **Justify your answer.**
(b) Suppose that f is continuous in K and analytic in a neighborhood of K. Is it true that f can be approximated (uniformly in K) by a sequence of polynomials? **Justify your answer.**
(c) Suppose that f is continuous in K and analytic in a neighborhood of K. Is it true that f can be approximated (uniformly in K) by a sequence of the following rational functions $R_n(z) = \sum_{j=0}^{m_n} a_j z^j + \sum_{k=1}^{l_n} \frac{b_k}{z^k}, n \in \mathbb{N}$? **Justify your answer.**