Math 542 **Comprehensive Examination** **January 2019**

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems. Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3, and 4. Each problem is worth 10 points. Justify all your answers. Good Luck!

Notation:
We denote the set of complex numbers by $\mathbb{C}$ and the set of integers by $\mathbb{Z}$. If D is a domain in $\mathbb{C}$, we denote by $A(D)$ the set of all functions $f : \overline{D} \to \mathbb{C}$ that are defined and continuous in $\overline{D}$ and analytic in D.

1. Find all complex solutions $z = x + iy$ of the inequality
$$|\cos z|^2 + |\sin z|^2 \leq 1.$$

2. Evaluate
$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)\, dx}{x(x-1)}.$$

3. Find a conformal mapping of the domain $D = \{z \in \mathbb{C} : |z| < 1,\ |z - i| < \sqrt{2}\}$ onto the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. (Hint: You can find the mapping as a composition of linear fractional transformations and powers of z. Carefully sketch all your intermediate domains, if any.)

4. Prove the following Hurwitz-type theorem. Let D be a domain in the complex plane. Let a sequence $f_n \in A(D)$ converge to $f \in A(D)$ uniformly on D as $n \to \infty$. Suppose that f has a zero in D, but f is not equal to zero identically. Prove that there is a number n such that the function f_n has a zero in D. (Hint: Use Rouché's theorem.)

5. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. Let $\mathcal{F} = \{f \in A(D) : \int_0^{2\pi} |f(e^{it})|^2\, dt \leq 1\}$. Is $\mathcal{F}$ a normal family in D?

6. Prove the identity that for all $z \in \mathbb{C} \setminus \mathbb{Z}$, we have
$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2}.$$