Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.
Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.
Justify all your answers. Good Luck!

<u>Notation:</u>
If $r > 0$, then D_r denotes $\{z \in \mathbb{C} : |z| < r\}$.
D denotes the open unit disk D_1.
$H(G)$ denotes the set of all analytic functions on an open set $G \subseteq \mathbb{C}$
If $K \neq \varnothing$ is not open, then $f \in H(K)$ means that f is analytic in a neighborhood of K

1. Let $G \subseteq \mathbb{C}$ be a bounded open non-empty connected set and let $f_1, f_2, \ldots, f_n$ be analytic in $\overline{G}$ (the closure of G). Prove that

$$\max_{z\in\overline{G}} \sum_{j=1}^{n} |f_j(z)| = \max_{z\in\partial G} \sum_{j=1}^{n} |f_j(z)| \ .$$

2. Prove that the equation

$$z \sin z = 1, z \in \mathbb{C}$$

has only real roots.

3. Let f be an entire function such that $|f(z)| \leq \sqrt{|z|}$ for sufficiently large z. Evaluate $f(2017)$. **Justify your answer.**

4. **Use residues** to calculate the following integral:

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2},$$

where $a > 1$. **Justify your answer.**

5. Let $\mathcal{F}$ be the class of functions $f \in H(D)$ such that

$$\iint\limits_{D_r} |f(z)|^2 dxdy \leq 1$$

for every $r \in (0, 1)$. Is this a normal family? **Justify your answer.**

6. Let

$$K = \left\{ z \in \mathbb{C} \; : \; |z| \leq 1 \text{ and } \left| z - \frac{1}{2} \right| \geq \frac{1}{2} \text{ and } \left| z + \frac{1}{2} \right| \geq \frac{1}{2} \right\}.$$

and let $f \in H(K)$.

(a) Is it true that f can be approximated uniformly on K by a sequence of polynomials? **Justify your answer.**

(b) Is it true that f can be approximated uniformly on K by a sequence of rational functions whose whose only poles lie in the set $\left\{-\frac{1}{2}, \frac{1}{2}, \infty\right\}$? **Justify your answer.**

(c) Let $G = \operatorname{int}(K)$ (the interior of K), and $f \in H(G)$. Is it true that f can be approximated uniformly on every compact subset of G by a sequence of polynomials? **Justify your answer.**