Math 542 Comprehensive Examination January 20 2016

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.
Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.
Justify all your answers. Good Luck!

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$
$H(G)$ *denotes the set of all analytic functions on an open set* $G \subseteq \mathbb{C}$

1. (a) Is there a Möbius transformation $w = T(z)$ such that $T(x) \in \mathbb{R}$ when $x \in \mathbb{R}$ and $T(i) = i$ and $T(-i) = -2i$? **Justify your answer.**
(b) Find a Möbius transformation $w = T(z)$ such that

$$T(i) = \infty, T(1-i) = i, T(2) = 1 - i.$$

Show your work.

2. Recall that a function $f(z)$ has a pole of order m at infinity if the function $g(\xi) = f\left(\frac{1}{\xi}\right)$ has a pole of order m at $\xi = 0$. Find all functions $f(z)$ which have in the extended complex plane only the following singularities: a pole of order n at $z = 0$ and a pole of order m at $z = \infty$. **Justify your answer.**

3. **Use residues** to evaluate the following definite integral (no other method will be accepted):

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}.$$

Justify your answer.

4. Let $f \in H(D), f : D \to D$.
(i) Prove that $|f''(0)| \leq 2$.
(ii) Find all $\{f \in H(D) : f(0) = f'(0) = 0\}$ and which satisfy $|f''(0)| = 2$ (your answer must be given in terms of an elementary function). **Justify your answer.**

5. Prove that for every $r > 0$ there is $N(r) \in \mathbb{N}$ such that for every $n \geq N(r)$, all zeros of the function

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + ... + \frac{1}{n!z^n}.$$

are contained in the disk $B(0, r) = \{z \in \mathbb{C} : |z| < r\}$

6. Let $G \subset \mathbb{C}$ be a non-empty open connected set and $\partial G \neq \varnothing$. Prove that there is $f \in H(G)$ such that f cannot be extended analytically to the boundary of G.