Math 542 Comprehensive Examination

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all seven problems.

Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3, 6 and 7.

Each problem is worth 10 points. Justify all claims.

$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ denotes the unit disc, $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ the upper half plane.

1. Find a positive constant R so that the power series
$$\sum_{k=0}^{\infty} \left(1 + \frac{1}{k}\right)^{k^3} z^{k^2}$$
converges for $|z| < R$ and diverges for $|z| > R$. Prove carefully that your answer is correct.

2. Let $0 < a < 1$. Evaluate the integral
$$\int_0^\infty \frac{x^{-a}}{1+x}\, dx$$
by contour integration. Show all estimates.

3. Find a conformal map from the semi-infinite strip $\{z = x + iy \colon x > 0, 0 < y < 1\}$ onto the half-disk $\{z = x + iy \colon |z| < 1, y > 0\}$.

4. Construct an entire function with simple zeros at the points $1, i, \sqrt{2}, 2i, \sqrt{3}, 3i, \ldots$ and no other zeros.

5. (a) State Montel's Theorem.

   (b) Let $\mathcal{F}$ be the family of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying the inequality
$$|f(z)| \leq \frac{1}{(1-|z|)^{2015}} \qquad \text{for all } z \in \mathbb{D}.$$
   Prove that $\mathcal{F}$ is a normal family.

6. Let f be an entire function such that
$$|f(z)| \leq 5|z|^{3/2} \qquad \text{for all } z \in \mathbb{C}, |z| \geq 1.$$
Prove that $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| + |b| \leq 5$.

7. (a) State the Argument Principle.

   (b) Apply the Argument Principle to find the number of solutions (counting multiplicities) to the equation $2z^4 - 2iz^3 + z^2 + 2iz - 1 = 0$ in $\mathbb{H}$. Justify your answer.