Math 542 Comprehensive Examination

January 24, 2014

Solve all eight problems. All problems have the same weight. Justify all claims.

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C}: |z| < 1\}$ in the complex plane and $\operatorname{Re} z$, $\operatorname{Im} z$ denote the real and respectively imaginary part of $z \in \mathbb{C}$.

1. Suppose that $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is an analytic function in $\mathbb{D}$. Prove that the series
$$\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k$$
converges in the whole plane and defines an analytic function F that satisfies
$$|F(z)| \leq Ce^{2|z|}$$
in the whole complex plane, for some positive constant C.

2. Let $\mathcal{F}$ be the family of all analytic functions f in $\mathbb{D}$ such that $f(\mathbb{D}) \subseteq \mathbb{D}$ and $f(1/2) = 0$. Find
$$\inf_{f \in \mathcal{F}} \operatorname{Im} f(0).$$

3. Find the number of roots of
$$z^4 + z^3 = 2z^2 - 2z - 4$$
in the first quadrant $\{z = x + iy : x \geq 0, y \geq 0\}$.

4. Using the Residue Theorem, evaluate
$$\int_0^{\infty} \frac{\log x}{x^2 + 2} dx.$$

5. Let f be a function that is analytic in $\mathbb{D}$ and continuous in the closure $\overline{\mathbb{D}}$. Assume also that $|f(z)| = 1$ for all z with $|z| = 1$. Prove that f is a rational function.

6. Suppose that D is a domain in the complex plane and (f_k) is a sequence of injective analytic functions in D that converges to a non-constant function f uniformly on compact subsets of D. Prove that f is injective.

7. Let z_0 be an arbitrary point in $\mathbb{D}$.

   (i) Prove that
$$\mathcal{F} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic in } \mathbb{D},\ f(z_0) = 1,\ \operatorname{Re} f(z) > 0 \text{ for all } z \in \mathbb{D}\}$$
   is a normal family.

   (ii) Let
$$\mathcal{H} = \{u : \mathbb{D} \to (0, \infty) : u \text{ is harmonic in } \mathbb{D},\ u(z_0) = 1\}.$$

Show that $\forall w \in \mathbb{D}$, $\exists M(w) \in (0, \infty)$ such that

$$\sup_{u \in \mathcal{H}} \left| \frac{\partial u}{\partial x}(w) \right| \leq M(w).$$

8. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lambda_n \neq 0$ for all n, and

$$\sum_{n=1}^{\infty} (1 - |\lambda_n|) < \infty.$$

Construct an analytic function f in $\mathbb{D}$ with zeros located exactly at λ_n's and with multiplicity according to the number of times λ_n appears in the sequence.