Math 542 Comprehensive Examination

January 2013

Each problem is worth 10 points. Justify all the claims that you make.

For $z \in \mathbb{C}$, let $\Re z$ and $\Im z$ denote the real and imaginary parts of z, respectively, so that $z = \Re z + i\Im z$. Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$, and let $\mathbb{H}$ denote the upper half-plane $\{z \in \mathbb{C} \colon \Im z > 0\}$.

1. Find all entire functions f that satisfy the inequality

$$|f(z)| \leq |z|^{3/2}, \quad \forall z \in \mathbb{C}.$$

2. Find a conformal map of the domain

$$D = \mathbb{D} \setminus \{z = \Re z + i0 \colon \Re z \in [0, 1)\},$$

obtained by removing the half-open interval $[0, 1)$ from the unit disc, onto the unit disc $\mathbb{D}$.

3. Find all entire functions whose set of zeroes coincides with the set of all non-negative integers, and so that each zero has order two.

4. Use the Residue Theorem to calculate the integral

$$\int_0^\infty \frac{x^{-1/6}}{x+1} dx.$$

5. How many solutions does the equation

$$z^4 - z^3 - 3z^2 + 8z + 2 = 0$$

have in the annulus $\{z \in \mathbb{C} \colon 1 < |z| < 3\}$?

6. Suppose that D is a domain in $\mathbb{C}$. Is there a sequence $(u_n)_{n\in\mathbb{N}}$ of harmonic functions in D that converges uniformly on compacta in D to the function $u(x, y) = x^3 - 2xy^2$?

7. Prove that if $f \colon \mathbb{H} \to \mathbb{H}$ is an analytic function and t is a positive real number, then $|f'(it)| \geq 1$ implies $\Im f(it) \geq t$.

8. Prove the following statement if true, or give a counterexample if it is false. If K is a compact subset of a domain $D \subseteq \mathbb{C}$ and f is an analytic function in D, then there exists a sequence of polynomials $(p_n(z))_{n\in\mathbb{N}}$ that converges to f uniformly on K.