Math 542, Comprehensive Examination
January 18, 2012

Solve all **eight** problems. Each problem is worth 10 points.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$, $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$.

1) Is there an analytic function $f \colon \mathbb{H} \to \mathbb{C}$ such that
$$\operatorname{Re} f(z) = x \arctan\left(\frac{y}{x}\right) + \frac{y \log(x^2+y^2)}{2} \quad \text{for all } z = x+iy \in \mathbb{H} \ ?$$
Justify your claim.

2) Evaluate the integral
$$\int_\gamma \frac{e^{-z}}{z^2-2} dz,$$
where γ is the imaginary axis with positive upward orientation.

3) Find a conformal map of $\mathbb{H} \setminus \{z = x+iy \colon x \geq 1,\ y = 0\}$ onto $\mathbb{H}$.

4) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of entire functions. Assume that this sequence converges to a polynomial f of degree $d \geq 1$ uniformly on compact subsets of $\mathbb{C}$.
   (i) Prove that there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$, the function f_n has at least d zeroes, counting multiplicity.
   (ii) Is it true that there must exist $N \in \mathbb{N}$ such that each f_n, $n \geq N$, has exactly d zeroes? Justify your claim.

5) Let f be an analytic function in $\mathbb{D}$, and assume that
$$\left| f\left(\frac{1}{n}\right) \right| \leq \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N},\ n \geq 2.$$
Prove that f vanishes identically in $\mathbb{D}$.

6) Let f be an analytic function in $\mathbb{D}$ with $|f(z)| \leq M$ for some $M > 0$ and all $z \in \mathbb{D}$. Prove that $|f'(1/2)| \leq 4M/3$. Is this bound sharp? Justify your claim.

7) Let $G(z)$ be defined by the infinite product
$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$
   (i) Show that $G(z)$ defines an entire function.
   (ii) Show that $\pi z G(z) G(-z) = \sin(\pi z)$ for all $z \in \mathbb{C}$.

8) Let $h(e^{i\theta})$ be a continuous function on the unit circle $\mathbb{T}$. Show that
$$\tilde{h}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} h(e^{i\varphi})\, d\varphi$$
defines a harmonic function in $\mathbb{D}$ and that $\lim\limits_{\substack{z \to z_0 \\ z \in \mathbb{D}}} \tilde{h}(z) = h(z_0)$ for all $z_0 \in \mathbb{T}$.