# Math 542 Comprehensive Examination

**January 19, 2011**

Solve **any eight** of the following **nine** problems.
$\mathbb{D}$ denotes the open unit disc disc in the complex plane $\mathbb{C}$.

1. Compute
$$I = \int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx,$$
justifying the relevant estimates. Here $a$ denotes a positive constant.

2. How many roots of the equation
$$z^4 + 3z - 5 = 0$$
are there in the annulus $A = \{z \in \mathbb{C}\colon 1 < |z| < 2\}$?

3. Find a conformal map of $\{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0,\ |z| < 1\}$ onto $\{z \in \mathbb{C}\colon 0 < \operatorname{Im}(z) < 1\}$.

4. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is an analytic function.
   (i) Prove that
$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}\, f(z_2)}\right| \leqslant \left|\frac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$
   (ii) Prove that
$$|f'(z_0)| \leqslant \frac{1 - |f(z_0)|^2}{1 - |z_0|^2} \quad \text{for all } z_0 \in \mathbb{D}.$$

5. Suppose that $f$ is an entire function and there $A, R_0 > 0$ positive constants and $m$ positive integer such that
$$|f(z)| \leqslant A|z|^m \quad \text{whenever } |z| \geqslant R_0.$$
Prove that $f$ is a polynomial of degree at most $m$.

6. Show that for each $\delta \in (0,1)$ there exists a constant $C = C(\delta) > 0$ such that, for every harmonic function $u : \mathbb{D} \to [-1,1]$,
$$\sup_{|z| \leqslant \delta} \left|\frac{\partial u}{\partial x}\right| \leqslant C.$$
Provide an explicit value for $C(\delta)$.

7. Let $f$ be a bounded analytic function on $\mathbb{D}$ with $f(0) \neq 0$. Assume that the set $(s_n)$ of zeros of $f$ (repeated with multiplicities) is infinite. Prove that

$$\sum_n (1 - |s_n|) \ < \ \infty.$$

8. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{D}$ such that

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

Find a sequence $(b_n)_{n=1}^{\infty}$ such that

$$\prod_{n=1}^{\infty} \left( b_n \frac{a_n - z}{1 - \bar{a}_n z} \right)$$

converges uniformly on compact sets in $\mathbb{D}$ to an analytic function $f$ whose only zeroes are $a_1, a_2, a_3, \ldots$.

9. Let $M > 0$. Prove that

$$\mathfrak{F} = \left\{ f : \mathbb{D} \to \mathbb{C} \colon f \text{ analytic}, \ \sup_{0 \leqslant r < 1} \int_0^{2\pi} |f(re^{it})| \, dt \leqslant M \right\}$$

is a normal family.