Solve any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. **Justify all your answers.** Good Luck!

Notation:
$$D = \{z \in \mathbb{C} \mid |z| < 1\}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$$
$H(G)$ denotes the set of all analytic functions on an open set $G \subset \mathbb{C}$

1. Let Ω be an open set in $\mathbb{C}$ containing the closed unit disk $\overline{D}$ and $f \in H(\Omega)$. Let Γ be a closed curve in $\mathbb{C}$ defined by $w(t) = f(e^{it}), 0 \le t \le 2\pi$. Prove that $\ell(\Gamma) \ge 2\pi |f'(0)|$, where $\ell(\Gamma)$ is the length of Γ.
   Hint: first prove that $\ell(\Gamma) = \int_0^{2\pi} |f'(e^{it})| \, dt$.

2. Let f be a meromorphic function in a neighborhood of $\overline{D}$ with zeros $z_1, z_2, ..., z_n \in D$ and poles $p_1, p_2, ..., p_m \in D$ (counted according to multiplicity) and analytic on the boundary of D. Suppose that $\text{Im}(f(z)) \ne 0$ for every z on the boundary of D. Prove that $m = n$.

3. Let P_n be a polynomial of degree n. For each $r > 0$, set
   $$M(r) = \max_{|z|=r}\{|P_n(z)|\}.$$

   (a) Prove that
   $$\frac{M(r_1)}{r_1^n} \ge \frac{M(r_2)}{r_2^n}$$
   whenever $0 < r_1 < r_2$.
   (b) Suppose that there are positive r_1 and r_2, $r_1 \ne r_2$, such that $\frac{M(r_1)}{r_1^n} = \frac{M(r_2)}{r_2^n}$. Prove that there is $a \in \mathbb{C}$ such that $P_n(z) = az^n$.

4. Let $\Omega \subseteq \mathbb{C}$ be a connected open set and let $(f_n)_{n=1}^{\infty}$ be a sequence of 1-1 analytic functions in Ω converging uniformly on each compact subset of Ω to a function f. Prove that f is either constant or 1-1 function in $H(\Omega)$.

5. Find the constants c_n, $n = 0, 1, 2, ...$, such that
   $$\frac{\pi}{\sin(\pi z)} = \frac{c_0}{z} + \sum_{n=1}^{\infty} \frac{c_n z}{z^2 - n^2}, z \in \mathbb{C} \backslash \mathbb{Z}$$

   and prove that the series $\frac{c_0}{z} + \sum_{n=1}^{\infty} \frac{c_n}{z^2-n^2}$ converges uniformly to $\frac{\pi}{\sin(\pi z)}$ on each compact subset of $\mathbb{C}$ after removing finitely many terms.
   Hint: Use Liouville's theorem to prove the equality; you can use without proof that the right-hand side of the above equality is periodic with period $p = 2$ and also use without proof the following inequality $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} \le \frac{\pi}{2a}, a > 0$.