Math 542 **Comprehensive Examination** **January 2009**

Do any four of the following five problems. Only four problems will be graded. Indicate clearly which problem is not to be graded. Each problem is worth 25 points. Justify all your answers. Good luck!

Notation:
C denotes the set of all complex numbers.

1. Let f be an analytic function in the unit disk such that $f(2^{-2n+1}) = f(2^{-2n})$ for all integers $n \geq 1$. Prove f is constant. (Hint. Compare $f(z)$ and $f(2z)$ and use a uniqueness theorem.)

2. Let f be an analytic function in the annulus $\{z \in \mathbf{C} : \frac{1}{2} < |z| < 2\}$. Suppose $\int_{|z|=1} f(z)z^n \, dz = 0$ for all integers $n \geq 0$. Prove f can be extended to be analytic in the whole disc $\{z \in \mathbf{C} : |z| < 2\}$.

3. How many zeros of the function $f(z) = z^3 + iz + 1000$ lie in the upper half-plane $\{z \in \mathbf{C} : \mathrm{Im} z > 0\}$? (Hint. Apply the argument principle to the boundary of a big semi-disk.)

4. Evaluate
$$\int\!\!\int_{|z|<1} \frac{dx\,dy}{z-w},$$
where $z = x + iy$ and $|w| < 1$. (Hint. Convert to polar coordinates r and θ and evaluate the integral in θ by the residue theorem.)

5. Prove that there is no conformal mapping that maps the punctured disk $\{z \in \mathbf{C} : 0 < |z| < 1\}$ onto the annulus $\{z \in \mathbf{C} : 2 < |z| < 3\}$.