

Do any four of the following five problems. Only four problems will be graded. Indicate clearly which problem is not to be graded. Each problem is worth 25 points. Justify all your answers. Good luck!

**Notation:**

$\mathbf{R}$  denotes the set of all real numbers.

$\mathbf{C}$  denotes the set of all complex numbers.

$H(D)$  denotes the set of all analytic functions on an open set  $D \subset \mathbf{C}$ .

1. Suppose  $f(z)$  has a pole at 0. Classify the singularity of  $e^{f(z)}$  at 0. Justify your answer.
2. (a) State the Runge approximation theorem.  
(b) Does there exist a sequence of polynomials  $p_n$  such that  $p_n(0) = 1$  and  $p_n(z) \rightarrow 0$  for every  $z \neq 0$ ? Justify your answer.
3. (a) Show there is a single-valued analytic branch  $f(z)$  of  $\sqrt{z^2 - 1}$  in  $\mathbf{C} \setminus [-1, 1]$  such that  $f(x) < 0$  for  $x > 1$ . Here  $[-1, 1]$  denotes a closed interval in  $\mathbf{R}$ .  
(b) Evaluate  $\int_{|z|=2} f(z) dz$  in counterclockwise direction, where  $f(z)$  is defined above in (a). (Hint. You can use the Laurent expansion of  $f(z)$  for big  $|z|$ .)  
*means  $|z|$  is a large number.*
4. Let  $D \subset \mathbf{C}$  be a disk of radius 1 with center at  $a \in \mathbf{C}$ , and let  $\partial D$  be the boundary of  $D$ . Prove that the following are equivalent:
  - (1)  $|a| < 1$ .
  - (2) There exists  $f \in H(D)$  continuous up to  $\partial D$  such that  $f(z) = 1/\bar{z}$  for  $z \in \partial D$ .
5. Let  $D = \{z \in \mathbf{C} : \text{Im}z > 0\}$  and  $\mathcal{F} = \{f \in H(D) : |f(z)| < 1, z \in D\}$ . Find  $\sup\{|f'(i)| : f \in \mathcal{F}\}$ . Is the supremum attained? Justify your answer.