University of Illinois at Urbana–Champaign
Department of Mathematics

Complex Analysis Comprehensive Examination (Math 542)
January 27, 2007

Solve any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. Show your work.

Notation. We denote the set of complex numbers by $\mathbb{C}$, and write $\mathbb{D} = \{\, z \in \mathbb{C} : |z| < 1 \,\}$ for the unit disk.

1. Determine all analytic functions f defined in $\mathbb{D}$ such that $f(z)^2 = \overline{f(z)}$ for all $z \in \mathbb{D}$. Prove that your answer is correct.

2. Let f be an analytic function defined in $\mathbb{D}$ such that $\operatorname{Re} f(z) > 0$ for all $z \in \mathbb{D}$ and such that $f(0) = \alpha > 0$. Prove that $|f'(0)| \le 2\alpha$.

3. Determine the numerical value of
$$\int_\gamma \frac{dz}{z^4 - 1},$$
where γ is the circle $\{\, z : |z - 2| = 2 \,\}$, covered once in the positive (counterclockwise) direction. Simplify your answer as much as possible. Show your work, so that it is clear what you did to calculate the answer. You should explain what steps you are taking, but you need not prove that your methods are correct.

4. We define
$$K = \{\, iy : y \ge 0 \,\} \cup \{\, x : x \ge 0 \,\} \cup \{\, e^{i\theta} : 3\pi/4 \le \theta \le 7\pi/4 \,\}$$
and $G = \mathbb{C} \setminus K$. Define $f(z) = (z^2 - \alpha z)/(z^2 - 1)$. If possible, give a specific numerical example of a complex number α with $\alpha \notin \{0, 1, -1\}$ such that there exists a function g analytic in G with $g(z)^2 = f(z)$ for all $z \in G$, and explain why your choice for α works. If there is no such α, explain why not.

5. Let D and D' be a simply connected plane domains, each different from the whole plane. Suppose that $z_1 \in D$ and $z_2 \in D'$. Let $\mathcal{F}$ be the family consisting of all functions f that are defined, analytic, and one-to-one in D, and satisfy $f(D) \subset D'$ and $f(z_1) = z_2$. Prove that $\mathcal{F}$ is a normal family.