Math 542 Comprehensive Examination January 21, 2006

Do any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. **Show your work.**

<u>*Notation:*</u>

$\mathbb{N} = \{0, 1, 2, 3, ...\}$
$B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$; $D = B(0, 1)$
$H(G)$ *denotes the set of all analytic functions on an open set* $G \subset \mathbb{C}$
$C(A)$ *denotes the set of all continuous functions on a subset* A *of* $\mathbb{C}$

1. (a) Let $h \in H(D) \cap C(\overline{D})$, $|h(z)| = 1$ when $|z| = 1$, and $h(z) \neq 0$ for every $z \in D$. Prove that $h(z) \equiv \alpha$ on D for some α with $|\alpha| = 1$.
(b) Find all entire functions f such that $|f(z)| = 1$ when $|z| = 1$.

2. Let γ be the ellipse $x^2 + y^2/4 = 1$ traversed once in the clockwise direction. Let

$$\operatorname{Log} z = \ln|z| + i\theta, \text{ where } \theta = \arg(z) \text{ and } -\pi < \theta \leq \pi, |z| > 0.$$

For an entire function f, evaluate the integral

$$\frac{1}{2\pi i} \oint_\gamma f'(z) \operatorname{Log} z\, dz.$$

(The answer involves the values of f at certain points.)

3. Let $f \in H(D)$ be such that $f^{-1}(0) = \{0\}$. Give a necessary and sufficient condition (in terms of the multiplicity of the zero) for the existence of a function $h \in H(D)$ with $h^2(z) = f(z)$, $z \in D$.

4. Prove for $0 < r < 1$ that the polynomial

$$P_n(z) = 1 + 2z + 3z^2 + ... + nz^{n-1}$$

has no zeros in $B(0, r)$ for sufficiently large n.

5. Let $\mathcal{F}$ be the family of all analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

on D such that $|a_n| \leq n$ for every n. Show that $\mathcal{F}$ is a normal family.