

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.

Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.

Justify all your answers. Good Luck!

1. Suppose f is analytic in the open unit disk $D \subseteq \mathbb{C}$ and f is continuous in the closed unit disk $\bar{D} \subseteq \mathbb{C}$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < 1$. If f has exactly m zeros in D (counting multiplicities in D), prove that

$$\min_{|z|=1} |f(z)| \leq |a_0| + |a_1| + \dots + |a_m| .$$

2. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$. Suppose that $f : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ is analytic and $a \in \mathbb{C}_+$. Prove that

$$|f'(a)| \leq \frac{\text{Im } f(a)}{\text{Im } a} .$$

3. Suppose $w = f(z)$ is meromorphic in \mathbb{C} and bounded in the annular domain $|z| > R$, $z \in \mathbb{C}$ for some $R > 0$. Prove that f is a rational function.

4. Use residues to evaluate the following integral:

$$\int_0^{\infty} \frac{\ln x}{(x+1)^2 \sqrt{x}} dx .$$

Justify your answer completely.

5. Let f be analytic in a neighborhood of the closed unit disk $\bar{D} \subseteq \mathbb{C}$ and f be injective on the unit circle ∂D . Prove that f is injective in the open unit disk $D \subseteq \mathbb{C}$.
6. Let D be the open unit disk in \mathbb{C} , $0 < r < 1$ and $\bar{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of analytic functions in D . Let $f : D \rightarrow \mathbb{C}$ be (at least) Riemann integrable over every closed disk \bar{D}_r , $0 < r < 1$ (i.e., $\text{Re } f$ and $\text{Im } f$ are Riemann integrable over \bar{D}_r). Suppose that the sequence $(f_n)_{n=1}^{\infty}$ converges to f in the L_1 -norm on \bar{D}_r , that is,

$$\|f_n - f\|_{1,r} = \iint_{\bar{D}_r} |f_n(x, y) - f(x, y)| dx dy \rightarrow 0 \text{ as } n \rightarrow \infty ,$$

for every $0 < r < 1$. Is it true that the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to the function f on every compact subset of D ? Justify your answer completely.