Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.
Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.
Justify all your answers. Good Luck!

Notation: *D denotes the unit disk in $\mathbb{C}$ and $H(G)$ denotes the set of all analytic functions on a non-empty set $G \subseteq \mathbb{C}$.*

1. Let $R > 0$ and let f be analytic in the open disk $|z| < R$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < R$ be its power series representation. Given $r \in (0, R)$, set $M(r) = \max_{|z|=r} |f(z)|$.
(a) Prove that $\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^2 d\varphi = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$ for every $r \in (0, R)$.
(b) **Use part (a)** to prove Cauchy's inequalities: $|a_n| \leq M(r)/r^n$ for every $r \in (0, R)$ and $n \in \mathbb{N} \cup \{0\}$.
(c) Prove that if one of Cauchy's inequalities becomes an equality, that is, there is $n_0 \in \mathbb{N} \cup \{0\}$ such that $|a_{n_0}| = M(r)/r^{n_0}$, then $f(z) = a_{n_0} z^{n_0}$ in the open disk $|z| < R$.

2. Let $R > 0$ and let f be analytic in the open disk $|z| < R$. Given $r \in (0, R)$, set $A(r) = \max_{|z|=r} \operatorname{Re}(f(z))$. Prove that $A : (0, R) \to \mathbb{R}$ is a strictly increasing function, unless f is constant.

3. Find the number of roots, counting multiplicities, of the following polynomial equation: $z^6 - 5z^4 + 8z - 1 = 0$ in the annular domain $\{z \in \mathbb{C} : 1 < |z| < 2\}$. **Justify your answer.**

4. **Use residues** to evaluate the following improper integral: $\int_0^{+\infty} \frac{x^a}{(1+x^2)^2} dx$, where $-1 < a < 3$ and $x^a = e^{a \ln x}$, $x > 0$. **Justify your answer** (the value of the integral alone is not sufficient; include derivation of auxiliary integrals when needed).

5. Let $G \subseteq \mathbb{C}$ be a non-empty open simply connected set and $z_1, z_2, ... z_n$ be distinct points in G. Set $E = G \setminus \{z_1, z_2, ..., z_n\}$ and, for every $j = j \in \{1, 2, ..., n\}$, denote by γ_j the circle $\gamma_j(t) = z_j + \rho_j e^{it}, t \in [0, 2\pi]$ where $\rho_j > 0$ is so small that the closed disk D_j centered at z_j of radius ρ_j is contained in G and $D_j \cap D_k = \varnothing$ if $j \neq k$. Let $f \in H(E)$. Set
$$a_j = \frac{1}{2\pi i} \int_{\gamma_j} f(z)\, dz \text{ and } h(z) = f(z) - \sum_{j=1}^{n} \frac{a_j}{z - z_j}.$$
Prove that h has an antiderivative in E.

6. Let $\mathfrak{F}$ be the family of all $f \in H(D)$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ such that $|a_n| \leq n^{2018}$ for every $n \in \mathbb{N}$. Is the family $\mathfrak{F}$ normal? **Justify your answer.**