Math 542 Comprehensive Examination August 17 2016

**Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.**
**Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems 1, 2, 3 and 4. Each problem is worth 10 points.**
**Justify all your answers. Good Luck!**

<u>Notation:</u>
$\mathbb{C}_\infty$ denotes extended complex plane $\mathbb{C} \cup \{\infty\}$ and $d_{\mathbb{C}_\infty}$ denotes the canonical metric on $\mathbb{C}_\infty$, i.e., the Euclidean distance between points in the Riemann sphere

$H(G)$ *denotes the set of all analytic functions on an open set* $G \subseteq \mathbb{C}$

$f_n \overset{K}{\rightrightarrows} f$ as $n \to \infty$ means that $(f_n)_{n=1}^{\infty}$ converges uniformly to $f$ on the set $K \subset \mathbb{C}$ in the metric $d_{\mathbb{C}_\infty}$ as $n \to \infty$

1. Use the Argument Principle to determine the number of roots, counting multiplicities, of
$$2z^5 + 6z^2 + z + 1 = 0$$
in the first quadrant.

2. Find a conformal mapping (analytic, one-to-one, onto) that maps the semi-disk $\{z \in \mathbb{C} : |z| < 1, x > 0\}$ onto the semi-infinite strip $\{z \in \mathbb{C} : x > 2,\ 0 < y < 2\}$.

3. Let $f$ be an analytic function in the annulus $\{z \in \mathbb{C} : \frac{1}{3} < |z - \frac{1}{3}| < 3\}$. Suppose $\int_{|z|=1} f(z) z^n dz = 0$ for all integers $n \geq 0$. Prove that $f$ may be extended to be analytic in the whole disc $\{z \in \mathbb{C} : |z - \frac{1}{3}| < 3\}$.

4. Let $f$ be a meromorphic function in $\mathbb{C}$ with a pole of order 5 at the origin 0. Classify the singularity of $e^{f(z)}$ at $z = 0$.

5. Give a meromorphic function on $\mathbb{C}$ which has simple poles at $z = ni$ with $\mathrm{Res}(f(z), ni) = n$ , for $n = 2, 4, 6, \ldots$, and has no other poles.

6. Let $G \subset \mathbb{C}$ be a non-empty open connected set and $(f_n)_{n=1}^{\infty}$ be a sequence in $H(G)$. Suppose that there is $f : G \to \mathbb{C}_\infty$ such that for every non-empty compact set $K \subset G$,
$$f_n \overset{K}{\rightrightarrows} f \text{ as } n \to \infty.$$
Prove that either $f \in H(G)$ or $f(z) = \infty$ for every $z \in G$.