Math 542 Comprehensive Examination August 2015

Students who are taking this exam as the Math 542 Graduate Comprehensive Exam should do all six problems.

Students taking this exam to satisfy the undergraduate complex analysis requirement should only do Problems I, II, III and IV.

Each problem is worth 10 points. Justify all claims.

I. Evaluate

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx.$$

Justify all estimates.

II. Let f be a rational function such that $|f(z)| = 1$ if $|z| = 1$.

(i) If f has no zeros or poles in $\{z : |z| < 1\}$, show that f is constant.

(ii) Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. Show that the number of zeros of f in $\{z : |z| < 1\}$ equals the number of poles of f in $\hat{\mathbb{C}} \setminus \{z : |z| \le 1\}$ with multiplicities taken into account.

III. Let $\Omega := \{z : |z| > 1\}$.

(i) Let $f(z) = \frac{z}{z-1}$. Show that $\frac{f'(z)}{f(z)}$ has an antiderivative on Ω.

(ii) Show that $\frac{z}{z-1}$ has an analytic logarithm on Ω. You may find the result in part (i) useful.

IV. (i) Find a one-to-one analytic mapping $w = f(z)$ from

$$\Omega := \{z : |z+i| < \sqrt{2}, |z-i| < \sqrt{2}\}$$

onto $\{w : |w| < 1\}$ such that $f(0) = 0$.

(ii) Is there a one-to-one analytic mapping $w = g(z)$ from Ω onto $\{w : |w| < 1\}$ such that $g(0) = 0$ and $g(1/2) = 9/10$? Justify your answer.

V. Give an example of an explicit meromorphic function on $\mathbb{C}$ having a pole at in with principal part

$$\frac{n^{5/2}}{(z-in)^2}$$

for $n = 1, 2, 3, \ldots$, and no other poles.

VI. Let $\mathcal{F}$ be the family of all analytic mappings f of $\{z : \operatorname{Re} z > 0\}$ into itself such that $f(1) = 1$. Show that there exists $g \in \mathcal{F}$ such that

$$|g'''(4)| = \sup_{f\in\mathcal{F}} |f'''(4)|.$$