Math 542 Comprehensive Examination

Solve all eight problems. Each problem is worth 10 points. Justify all claims. Denote $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$, $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, $\mathbb{Z} =$ the set of integers, and $A(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \colon f \text{ analytic}\}$.

1. (a) Are there nonconstant bounded analytic function on $\mathbb{C} \setminus \mathbb{Z}$?
   (b) Are there analytic functions $f \colon \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ and $f(1/2) = 3/4$?
   Prove your answers.

2. Find explicit conformal maps:
   (a) between the half-strip $\{z = x + iy \colon 0 < x < \pi/2, y > 0\}$ and $\mathbb{H}$.
   (b) between the punctured disc $\mathbb{D} \setminus \{0\}$ and the slitted plane $\mathbb{C} \setminus [-2, 2]$.

3. Does the function $f(z) = \frac{1}{1-z^2}$ have an antiderivative in $\mathbb{C} \setminus \{1, -1\}$? Prove your answer.

4. (a) Prove that a power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $1/a$, where $a = \limsup_{n\to\infty} |c_n|^{1/n}$.
   (b) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} (\sin n)^{3n} z^n$. Justify your answer.

5. Let $f \colon \mathbb{D} \to \mathbb{D}$ be analytic, with $f(0) = 0$ and $-1 < \operatorname{Re} f(z) < 1$ for all $z \in \mathbb{D}$. Prove that

$$|\operatorname{Im} f(z)| < \frac{2}{\pi} \log \frac{1+|z|}{1-|z|}, \qquad z \in \mathbb{D}.$$

6. Use the method of residues to calculate the (improper Riemann) integral

$$\int_0^{\infty} \frac{\sin x}{x}\, dx.$$

   Show all estimates.

7. (a) Suppose that $f \in A(\mathbb{D})$ and $a \in \mathbb{D}$, $R > 0$ are such that $\{z \in \mathbb{C} \colon |z - a| \leqslant R\} \subseteq \mathbb{D}$. Prove that

$$|f(a)|^2 \leqslant \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{it})|^2 r\, dr dt.$$

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(b) Which of the following two families of analytic functions on $\mathbb{D}$ is normal?

$$\mathcal{F}_1 = \left\{ f \in A(\mathbb{D}) : \iint_{\mathbb{D}} |f(x+iy)|^2 dxdy \leqslant 1 \right\}.$$

$$\mathcal{F}_2 = \left\{ f \in A(\mathbb{D}) : \iint_{\mathbb{D}} |f(x+iy)|^2 dxdy < \infty \right\}.$$

Justify your answer.

8. Find an entire function whose zero set (with multiplicities) is exactly

$$\{\ln 1, \ln 2, \ln 2, \ln 3, \ln 3, \ln 3, \ln 4, \ln 4, \ln 4, \ln 4, \ldots\}.$$

Justify your answer.