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Complex Analysis Examination

Each problem is worth 10 points. Justify all the claims that you make.

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$.

1. Evaluate and simplify the following
   (a) $|\mathrm{Log}(e + ie)|$; (b) principal value of i^i; (c) $\sin(\frac{\pi}{2} + \frac{\pi}{2}i)$.
2. Find the Laurent series of $\frac{1}{z^2-1}$ in powers of $z-2$ in the region $1 < |z-2| < 3$.
3. State and prove the maximum modulus principle for functions that are continuous on the closed disk $\overline{\mathbb{D}}$ and analytic in the open disk $\mathbb{D}$.
4. Does $f(z) = \frac{1}{z-1} - \frac{1}{z+1}$ have an antiderivative in $|z| > 2$?
5. Find the image of the semi-infinite strip $\{z = x + iy\colon\ -\frac{\pi}{2} \le x \le \frac{\pi}{2},\ y \ge 1\}$ under the mapping $f(z) = \sin z$.
6. Find the fractional linear transformation T that maps the unit disk $\mathbb{D}$ onto the half plane $\{z = x + iy\colon x > 1\}$ with $T(0) = 2$ and $T(1) = 1$.
7. Let C be the unit circle $|z| = 1$ oriented counterclockwise. Evaluate
$$\int_C \left(\frac{1}{\sin z} + \frac{\cos z}{(z - \frac{\pi}{4})^4} \right) dz\,.$$
8. Evaluate
$$\int_0^\infty \frac{\cos x}{x^4 + 16} dx\,.$$