Math 542 Comprehensive Examination

August 20, 2012

$\mathbb{D}$ denotes the open unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbb{T}$ the unit circle $\{z \in \mathbb{C}\colon |z| = 1\}$.

Solve **any eight** of the following **nine** problems. Each problem is worth 10 points.

1. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{z^{n^3}}{n^4}.$$

   Justify your claim.

2. Use the Residue Theorem to calculate the integral

$$\int_0^{\infty} \frac{x^{3/2}}{1+x^3}\,dx.$$

   Justify all estimates.

3. Find an explicit conformal mapping of $\mathbb{C} \setminus [0,1]$ onto $\mathbb{D} \setminus \{0\}$.

4. Let f be an analytic function in $S = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ for some $a < b$. We suppose that

$$|f(z)| \leq \frac{C}{|z|^{\alpha}}$$

   for all $z \in S$, some positive constant C, and a constant $\alpha > 1$. Prove that the function g defined by

$$g(x) = \int_{-\infty}^{\infty} f(x+iy)dy$$

   is constant in (a,b).

   Is it enough to requite that α is positive? Explain.

5. Find the family of all functions f analytic in $\mathbb{D}$ and continuous on the closed unit disc $\overline{\mathbb{D}}$, such that

$$|f(z)| = e^{\operatorname{Re} z}$$

   for all $z \in \mathbb{T}$.

6. Prove that if non-constant real-valued functions u, v, and their product uv are harmonic in a domain D, then there exists a real constant c such that the function $u + icv$ is analytic in D.

   Hint. Consider the function f/g, where $f = u_x - iu_y$ and $g = v_x - iv_y$.

7. Suppose a function $f : \mathbb{D} \to \mathbb{C}$ is analytic, bounded, and non-constant. Let (s_n) be the zeros of f counted with multiplicities. Show that

$$\sum_n (1 - |s_n|) < \infty.$$

8. (i) (7 points) Construct an entire function with zeros located exactly at $s_n = \sqrt{n}$, $n \in \mathbb{N}$, and with multiplicity n^2. Justify your answer.
(ii) (3 points) What is the general form of such a function? Justify your answer.

9. (i) (5 points) State Runge's approximation theorem.
(ii) (5 points) Let A be an open annulus. Show that there exists a function f, analytic in A and with the property that there is no sequence (P_n) of analytic polynomials with $\sup_K |f - P_n| \to 0$ for any compact set $K \subset A$.