Math 542 Comprehensive Examination

August 18, 2011

Solve **any eight** of the following **nine** problems. Each problem is worth 10 points.

Below, $B(z_0, r)$ denotes the open disc centered at $z_0 \in \mathbb{C}$ of radius $r > 0$.

1. Evaluate the integral
$$I = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$
Justify the estimates that you are using.

2. Let f be an analytic function on $\mathbb{C}$. Assume that there exists $n \in \mathbb{N}$ and a function $\phi : (0, \infty) \to [0, \infty)$ such that
$$\lim_{r\to\infty} \frac{\phi(r)}{r^n} = 0,$$
and
$$\forall r > 0, \forall z \in B(0, r), \quad 0 < |f(z)| \leqslant \phi(r).$$
Prove that f is a constant function.

3. Let $d, n \in \mathbb{N}$, let $N(n)$ denote the number of solutions of the equation
$$2nz^d + nz + 1 = 0$$
in $B(0, 1)$. Find
$$\lim_{n\to\infty} N(n).$$
Justify your claim.

4. Prove that every conformal map of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto itself can be expressed in the form
$$z \mapsto \frac{az + b}{cz + d},$$
for some $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

5. Let $D \subseteq \mathbb{C}$ be a domain and $z_0 \in \mathbb{C}$. Assume that (f_n) is a sequence of analytic functions in D such that $\lim_{n\to\infty} f_n(z_0) = w_0 \in \mathbb{C}$ and the sequence of derivatives (f_n') converges uniformly on compact subsets of D to a function g.
Is it true that there exists an analytic function f in D such that $f_n \to f$ uniformly on compact subsets of D? Prove or give a counterexample.

6. Prove that
$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}, \qquad z \in \mathbb{C}.$$

7. Let $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $\mathcal{F} = \{f : H \to H : f \text{ analytic}, f(1) = 1\}$.
   (i) Show that $\mathcal{F}$ is a normal family.
   (ii) Show that there exists $g \in \mathcal{F}$ with $|g'''(4)| = \sup_{f\in\mathcal{F}} |f'''(4)|$.

8. (i) Let (u_n) be a sequence of harmonic functions in a domain D such that $u_n \to u$ uniformly on compact subsets of D. Prove that u is harmonic in D.
   (ii) Show that if u is harmonic in D, then its partial derivatives u_x and u_y are harmonic in D.

9. Let A_r denote the annulus $A_r = \{z \in \mathbb{C} : 1 < |z| < r\}$, $r > 1$. Show that if the annuli A_{r_1} and A_{r_2} are conformal, then $r_1 = r_2$.