Math 542 Comprehensive Examination August 2010

Solve any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. **Justify all your answers.** Good Luck!

<u>*Notation:*</u>
$D = \{z \in \mathbb{C} \mid |z| < 1\}$
$B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
$H(G)$ denotes the set of all analytic functions on an open set $G \subset \mathbb{C}$

1. How many zeros does the polynomial

$$f(z) = iz^5 + z - 2010$$

have in the upper half-plane $\operatorname{Im} z > 0$? Does it have any multiple zeros?
<u>Hint</u>: *apply the argument principle to a large semi-disk.*

2. Let f be a continuously differentiable function on the unit circle $\partial D = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $\gamma(t) = e^{it}, 0 \le t \le 2\pi$. Define

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)\, dw}{w - z}, \quad z \in D.$$

Prove that F is bounded in D.
<u>Hint</u>: *consider the point $z_0 \in \partial D$ which is the closest point to z.*

3. (a) Prove that there exists an analytic function in the domain $\Omega = \{z \in \mathbb{C} \mid |z| > 1\}$ such that

$$[f(z)]^2 = z^2 + z, \; f(2) = -\sqrt{6}.$$

(b) For f in part (a), evaluate $\int_\Gamma f\, dz$, where $\Gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$.
<u>Hint:</u> *use the Laurent series of f.*

4. Let $f, g \in H(D)$. Suppose $|f(z)| \ge |g(z)|$ for all $z \in D$. Define $E = \{z \in D \mid |f(z)| = |g(z)|\}$. Prove that either $E = D$ or E has no limit points in D.

5. Let $\Omega \subseteq \mathbb{C}$ be an open connected set and let $(f_n)_{n=1}^\infty \subseteq H(\Omega)$. Suppose that the sequence $(f_n)_{n=1}^\infty$ is locally uniformly bounded, that is, for every $z \in \Omega$, there is $M_z > 0$ and $\varepsilon_z > 0$ such that $\sup_{\xi \in \Omega \cap B(z, \varepsilon_z)} |f_n(\xi)| \le M_z$ for every $n = 1, 2, \ldots$. Also, suppose that the pointwise limit $\lim_{n \to \infty} f_n(z)$ exists for z in a subset A of Ω with a limit point in Ω. Prove that for every compact $K \subseteq \Omega$, the sequence $(f_n)_{n=1}^\infty$ converges uniformly on K.