Math 542 **Comprehensive Examination** **August 2009**

Please do any four of the following five problems. Only four problems will be graded. Indicate clearly which problem is not to be graded. Justify all your answers. Good luck!

1. Does there exist a sequence of polynomials $P_n(z)$ such that

$$\lim_{n\to\infty} P_n(z) = \begin{cases} 1 & \text{if} \quad \mathrm{Re}\, z > 0 \\ 2 & \text{if} \quad \mathrm{Re}\, z = 0 \\ 3 & \text{if} \quad \mathrm{Re}\, z < 0 \end{cases} \qquad ?$$

   Prove your answer.

2. Let $D = \{z : |z| < 1\}$, and f be an analytic function on D with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$. Prove that
   (1) $|f(z)| \leq |z|$ for all $z \in D$;
   (2) if $|f(z_0)| = |z_0|$ for some $z_0 \in D$, $z_0 \neq 0$, then $f(z) = cz$ for all $z \in D$, where c is a constant.

   Please do not quote the Schwarz Lemma.

3. Let $a > 1$. How many solutions does the equation

$$a - z - e^{-z} = 0$$

   have in the half plane $\{z : \mathrm{Re}\, z > 0\}$? Justify your answer.

4. Use contour integration to calculate

$$\int_0^\infty \frac{\sin x}{x} dx.$$

   Give detailed explanations on all claims.

5. Find the constants c_n so that

$$\frac{1}{\sin^2 z} = \sum_{n=-\infty}^{+\infty} \frac{c_n}{(z - \pi n)^2}$$

   and the series converges uniformly on every bounded set after dropping finitely many terms. Justify all your claims. (Hint: Use Liouville's theorem to prove the equality.)