Math 542 **Comprehensive Examination** **August 2008**

Do any four of the following five problems. Only four problems will be graded. Indicate clearly which problem is not to be graded. Each problem is worth 25 points. Justify all your answers. Good luck!

Notation:
R denotes the set of all real numbers.
C denotes the set of all complex numbers.

1. Find all entire functions $f(z)$ such that $\int\int_{\mathbf{C}} |f(z)|\, dx\, dy = 1$, where $z = x+iy$. Justify your answer. (Hint: estimate the Taylor coefficients of f.)

2. Put $\Omega = \{z \in \mathbf{C} : |\mathrm{Re} z| < \pi/4, \mathrm{Im} z > 0\}$. Find the image $f(\Omega)$ if

   (a) $f(z) = \sin z$.

   (b) $f(z) = (z + \frac{i\pi}{4})^{-1}(z - \frac{i\pi}{4})$.

3. (a) Show there exists a single-valued analytic branch $f(z)$ of $\sqrt[3]{\frac{z-1}{z+1}}$ in $\mathbf{C}\setminus[-1,1]$ such that $f(x) > 0$ for $x > 1$. Here $[-1,1]$ denotes a closed interval in $\mathbf{R}$.

   (b) Evaluate $\int_{|z|=2} f(z)\, dz$ in counterclockwise direction, where $f(z)$ is defined above in (a). (Hint. You can use the Laurent expansion of $f(z)$.)

4. Let f be a meromorphic function in $\mathbf{C}$ and let γ be a simple closed curve in $\mathbf{C}$ on which f does not take the values 0 or ∞. Prove

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = N - P,$$

   where N and P are respectively the numbers of zeros and poles of f inside γ counting multiplicities.

5. Let $f(z) = e^{1/z}$. Find the domain of convergence and the sum of the series

$$\sum_{n=0}^{\infty} \frac{2^n f^{(n)}(z)}{n!}$$