Department of Mathematics

University of Illinois at Urbana–Champaign

Math 542   Complex Analysis

Comprehensive Examination   August 2007

Solve any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. **Show your work.**

1. Let f be an analytic function of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ into itself with $f(0) = 0$ and $f'(0) = 0$. Prove that for all $z \in \mathbb{D}$ we have $|f(z)| \leq |z|^2$. Make sure that you justify all steps in your proof.

2. Determine the number of zeros, counting multiplicities, of the polynomial

$$f(z) = z^{10} - 2\pi z^7 + iez^3$$

   in the unit disk $\mathbb{D}$. Prove that your answer is correct.

3. Let f be analytic in $\mathbb{D}$, satisfying $|f(z)| \leq 1/(1 - |z|)$ for all $|z| < 1$, and let $|a| < 1$. Prove that for each integer n there is a constant $C(n)$ so that

$$|f^{(n)}(a)| < \frac{C(n)}{(1 - |a|)^{n+1}}.$$

   The constant $C(n)$ may depend on n, but should not depend on f or a.

4. Evaluate the following integral by using the Cauchy residue theorem:

$$\int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^2}\, dx.$$

   Justify each step and all estimates used.

5. Let f be the conformal mapping of $\{z \in \overline{\mathbb{C}} : |z - 1| > 1\}$ onto $\{z \in \mathbb{C} : \operatorname{Im} z > -2\}$ such that $f(3) = 0$ and $-if'(3) > 0$. Here $\mathbb{C}$ denotes the complex plane and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Find the numerical value of $f(-6)$. Simplify your answer as much as possible.