Math 542 **Comprehensive Examination** **August 22, 2006**

Do any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. **Show your work.**

Notation: $B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$
$H(G)$ denotes the set of all analytic functions on an open set $G \subset \mathbb{C}$

1. Find the largest open set $\Omega \subset \mathbb{C}$ on which

$$\Pi_{n=1}^{\infty} \frac{1-z^n}{1+z^{2n}}$$

converges.

2. Let $\Omega = \{re^{i\theta} : 0 < r < \infty, |\theta| < \pi/4\}$.
(i) Show that there exists an analytic mapping $g : \Omega \to B(0,1)$ such that $g(1) = 0$ and $g(2) = 1/2$.
(ii) Show that there does not exist an analytic mapping $h : \Omega \to B(0,1)$ such that $h(1) = 0$ and $h(2) = 3/4$.

3. Let $\mathcal{F}$ be the family of all analytic functions on $B(0,1)$ such that

$$\sup_{0 \le r < 1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right| d\theta \le M,$$

for some $0 < M < +\infty$. Show that $\mathcal{F}$ is a normal family.

4. Let $\Omega = \{x+iy : |x|+|y| < 2\} \setminus \{x+iy : |x| \le 1 \text{ and } |y| \le 1\}$. Suppose $f \in H(\Omega)$.
(i) Must there exist polynomials Q_n such that $Q_n \to f$ locally uniformly on Ω?
(ii) Must there exist polynomials Q_n such that $Q_n \to f$ uniformly on Ω?

5. Suppose f is analytic and not a constant on $\{z : 0 < |z| < 3/2\}$. Show that f has an essential singularity at $z = 0$ if and only if for every positive integer k, there is $z_n \to 0$ such that $|f(z_n)| < |z_n|^k$.