Math 540 Comprehensive Examination
May 16, 2017

Do five out of six problems. Each problem is worth 20 points. Justify all claims.

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space (i.e. $\mu(X) < \infty$), let $\alpha > 0$, and let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be such that $\mu(A_n) \geq \alpha$ for each $n \in \mathbb{N}$. Put

$$A := \{x \in X : \exists^\infty n \; x \in A_n\},$$

where $\exists^\infty$ means "for infinitely many $n \in \mathbb{N}$".

(a) Show that $A \in \mathcal{M}$.

(b) Prove that $\mu(A) \geq \alpha$.

(c) Give an example of a measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) = \infty$ and a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that $\mu(A_n) \geq 1$ for each $n \in \mathbb{N}$, but $A = \emptyset$.

2. Compute

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+e^{-kx}} \frac{1}{1+x^2}\, dx.$$

Justify your computation.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and, for each $n \in \mathbb{N}$, let $f_n, f : X \to [0, \infty]$ be non-negative measurable functions. Suppose that $f_n \to_\mu f$ (i.e. converges in measure).

(a) Prove that, for each subsequence $(n_k)_k$, there is a further subsequence $(n_{k_\ell})_\ell$ such that

$$\int f d\mu \leq \liminf_{\ell\to\infty} \int f_{n_{k_\ell}} d\mu.$$

(b) Deduce that $\displaystyle\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on it. Call $(f_n)_{n\in\mathbb{N}}$ *uniformly absolutely continuous in* L^1 if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable $A \subseteq X$, $\mu(A) < \delta \implies \forall n \in \mathbb{N} \int_A |f_n| d\mu < \varepsilon$. Say that $(f_n)_{n\in\mathbb{N}}$ *uniformly vanishes at* ∞ *in* L^1 if for every $\varepsilon > 0$ there is a measurable $A \subseteq X$ with $\mu(A) < \infty$ such that $\forall n \in \mathbb{N} \int_{A^c} |f_n| d\mu < \varepsilon$.

4. Let $(f_n)_n \subseteq L^1(X, \mathcal{M}, \mu)$. Assume that $(f_n)_n$ converges to f in measure, that $(f_n)_n$ is uniformly absolutely continuous in L^1, and that $(f_n)_n$ uniformly vanishes at ∞ in L^1. Prove that $(f_n)_n$ converges to f in L^1.

5. Let $p \in (1, \infty)$ and let q be its Hölder conjugate exponent. Prove that for any $f \in L^p(\mathbb{R}, \lambda)$ and $g \in L^q(\mathbb{R}, \lambda)$, $f * g$ is uniformly continuous and $(f * g)(x) \to 0$ as $|x| \to \infty$.

6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $K : X \times X \to \mathbb{R}$ be nonnegative and measurable with respect to $\mathcal{M} \otimes \mathcal{M}$. Suppose there exists a positive $M \in \mathbb{R}$ such that

$$\int_X K(x,y)\, d\mu(y) \leq M \text{ and } \int_X K(x,y)\, d\mu(x) \leq M$$

for each $x \in X$. For a measurable $f : X \to \mathbb{R}$ and $x \in X$, put

$$Tf(x) = \int_X K(x,y) f(y)\, d\mu(y)$$

if the integral exists.

(a) Fix x such that $Tf(x)$ exists. Prove that

$$\left| \int_X K(x,y) f(y)\, d\mu(y) \right| \leq M^{1/q} \left(\int_X K(x,y) |f(y)|^p\, d\mu(y) \right)^{1/p}$$

where q denotes the Hölder conjugate exponent to p.

(b) Prove that for any $p \in (1, \infty)$ and $f \in L^p(X, \mathcal{M}, \mu)$, $\|Tf\|_p \leq M\|f\|_p$.