Math 540 Comprehensive Examination May 19, 2016

Solve five of the following six problems. Each problem is worth 20 points. Calculators, books and notes are not allowed. Good Luck!

<u>Notation:</u> m is the Lebesgue measure on $\mathbb{R}$
If E is a non-empty Lebesgue measurable set in $\mathbb{R}$ and $p \in [1, +\infty]$, then $L^p(E)$ denotes Lebesgue's L^p-space

1. Suppose f_n, f, g_n and g are Lebesgue measurable functions on a non-empty Lebesgue measurable set $E \subseteq \mathbb{R}$. Let $f_n \to f$ as $n \to \infty$ in Lebesgue's measure m and $g_n \to g$ as $n \to \infty$ in Lebesgue's measure m.
(a) Prove that $f_n g_n \to fg$ as $n \to \infty$ in Lebesgue's measure m if $m(E) < \infty$.
(b) Without the condition $m(E) < \infty$, is part (a) still true? **Justify your answer.**

2. Let f be a continuous function on $[0, 1]$. Suppose that f is absolutely continuous on the closed interval $[\epsilon, 1]$ for every $\epsilon > 0$.
(a) Is f absolutely continuous on $[0, 1]$? **Justify your answer.**
(b) Assume that f is increasing on $[0, 1]$. Is f absolutely continuous on $[0, 1]$? **Justify your answer.**

3. For $1 \leq p \leq \infty$. Let $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$. Here $\frac{1}{p} + \frac{1}{p'} = 1$, where we agree that if $p = 1$, then $p' = +\infty$ and if $p = +\infty$, then $p' = 1$. Recall that the *convolution* of f and g is defined by $(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) dy$.
(a) Is $f * g$ bounded? **Justify your answer.**
(b) Is $f * g$ continuous on $\mathbb{R}$? **Justify your answer.**

4. Let (M, ρ) be a metric space and let $\eta^* : 2^X \to [0, +\infty]$ be an outer measure on M. The outer measure η^* is called the *metric measure* if the following condition is satisfied: if $E, F \subseteq M$, $E \neq \varnothing$, $F \neq \varnothing$ and $\operatorname{dist}(E, F) = \inf_{x \in E, y \in F} \rho(x, y) > 0$, then $\eta^*(E \cup F) = \eta^*(E) + \eta^*(F)$. Let $\nu^* : M \to [0, +\infty]$ be an outer measure on the metric space (M, ρ) such that every open set in (M, ρ) is ν^*-measurable in the sense of Carathéodory. Prove that ν^* is the metric outer measure on M.

5. Let $(f_n : \mathbb{R} \to [0, +\infty))_{n=1}^{\infty}$ be a sequence in $L^1(\mathbb{R})$ such that $f_n(x) \to f(x)$ a.e. in $\mathbb{R}$, where $f \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} f_n dx \to \int_{\mathbb{R}} f dx$ as $n \to \infty$. Prove that for every non-empty Lebesgue measurable set $E \subseteq \mathbb{R}$,
$$\int_E f_n(x)\, dx \to \int_E f(x)\, dx \text{ as } n \to \infty .$$

6. Let $E \subseteq \mathbb{R}$ be a non-empty Lebesgue measurable set of finite Lebesgue's measure μ and let $f : E \to [0, +\infty)$ be a Lebesgue measurable function. Prove that $f \in L^1(E)$ if and only if the infinite series
$$\sum_{n=0}^{\infty} 2^n \mu(\{x \in E : f(x) \geq 2^n\})$$
converges.