Math 540 Comprehensive Examination, May 2019

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Let $\epsilon > 0$ and a, q be relatively prime natural numbers. Define (a,q)-type set $E_{a/q}$ by
$$E_{a/q} = \{x \in \mathbb{R} : |x - \tfrac{a}{q}| \leq \tfrac{1}{q^{2+\epsilon}}\}.$$
Show that those points in $\mathbb{R}$ belonging to infinitely many (a,q)-type sets form a zero Lebesgue measure set.

2. Let $\phi_n : [0,1] \to \mathbb{R}$ be a sequence of nondecreasing functions so that $\sum_{n=1}^{\infty} \phi_n(0)$ and $\sum_{n=1}^{\infty} \phi_n(1)$ are convergent series. Show that
i) The series $f(x) = \sum_n \phi_n(x)$ is a well-defined measurable function on $[0,1]$, and
ii) for m *a.e.* $x \in [0,1]$, the series $\sum_n \phi_n'(x)$ converges to $f'(x)$.

3. Prove that
$$\lim_{n\to\infty} \int_0^1 e^{int^2} dt = 0.$$

4. Let $1 \leq p < \infty$. Suppose that $\{f_n\}$ be a sequence of real valued measurable functions on a measure space $(X, \mathcal{B}, \mu)$, which satisfies
$$\left\| f_{n+1} - f_n \right\|_{L^p(X)} \leq \tfrac{1}{2^n}$$
for all $n \in \mathbb{N}$. Prove that $\{f_n\}$ converges to a function, $f \in L^p(X)$, μ almost everywhere.

5. Let $f_1, f_2, f_3 \in L^{\frac{3}{2}}(\mathbb{R}, m)$. Prove that
$$\int_{\mathbb{R}\times\mathbb{R}} \left| f_1(x_1) f_2(x_2) f_3(x_1 + x_2) \right| dx_1 dx_2 \leq \prod_{j=1}^{3} \|f_j\|_{L^{\frac{3}{2}}}.$$

6. Let $(X, \mathcal{M}, \mu)$ be a <u>finite</u> measure space. Given measurable $f : X \to \mathbb{C}$, let E_f denote its distribution function:
$$E_f(\lambda) = \mu(\{x : |f(x)| > \lambda\}), \quad \lambda > 0.$$
Assume that $f_n : X \to \mathbb{C}$ is a sequence of measurable functions satisfying
i) $f_n \to f$ μ *a.e.*,
ii) there exists a Lebesgue measurable function $E : (0, \infty) \to [0, \infty)$ with $\int_0^\infty E(\lambda) dm(\lambda) < \infty$ such that $E_{f_n}(\lambda) \leq E(\lambda)$ for each n and $\lambda > 0$.
Prove that $f_n \to f$ in L^1.
Hint: It is crucial that μ is a finite measure.