Math 540 Comprehensive Examination

May 22, 2015

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m. Calculators, books and notes are not allowed.

1. Suppose that $f_n : X \to [0, \infty]$ is measurable for any $n \in \mathbb{N}$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$.
For each statement, give a counterexample or a short proof/explanation.
a) $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.
b) if $f_1 \in L^1(\mu)$, then a) holds.

2. Let $1 < p < \infty$, $f \in L^p((0, \infty))$ and define

$$Tf(x) = \frac{1}{x} \int_0^x f(t) dm,$$

for $x \in (0, \infty)$. Here m is Lebesgue measure.
Prove Hardy's inequality

$$\|Tf\|_p \leq \frac{p}{p-1} \|f\|_p.$$

and the equality holds if and only if $f = 0$ a.e.

3. Suppose that μ is a measure on X with $\mu(X) < \infty$, $f_n \in L^1(\mu)$, and $f_n(x) \to f(x)$ a.e. There exists $p > 1$ and a constant C such that

$$\sup_{n\in\mathbb{N}} \int_X |f_n|^p d\mu \leq C.$$

Prove that (f_n) converges to f in $L^1(\mu)$.

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Fix $p > 0$, and suppose that a sequence E_n of measurable subsets satisfies

$$\sum_n [\mu(E_n)]^p < \infty.$$

i) Prove that $\mu(\limsup E_n) = 0$ provided that $p \leq 1$,
ii) Give a counterexample to the statement in part i, when $p > 1$.

5. Prove or give a counterexample: If $f \in L^1(\mathbb{R}, m)$, then

$$\text{esssup}_{x\in I} |f(x)| < \infty$$

for some open interval I.

6. Let f be a function on $[a, b]$ of total variation $T_a^b f < \infty$.

(i) Prove that $\int_{[a,b]} |f'| \leqslant T_a^b f$.

(ii) Prove that if f is absolutely continuous then equality holds in (i).