Math 540 Comprehensive Examination

May 20, 2014

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Let m^* be defined by, for any $E \subset \mathbb{R}$,

$$m^*(E) = \sup\{m(K) : K \subset E \text{ and } K \text{ is closed}\}.$$

Here m is Lebesgue measure. Show that there exists a Lebesgue measurable set F such that $m(F) = m^*(E)$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space. For each statement give a counterexample or a proof/explanation.

a) $f_j \to f$ a.e. and $\sup_j \|f_j\|_p \leq 1$ for some $p > 1$, then f_j converges to f in L^1.

b) $f_j \to f$ a.e., $\sup_j \|f_j\|_p \leq 1$ for some $p > 1$ and $\mu(X) < \infty$, then f_j converges to f in L^1.

3. Let μ be a measure on X, $0 < p < \infty$, $f \in L^p$. Suppose that (f_j) is a sequence of L^p functions such that $f_j \to f$ a.e. and $\lim_{j\to\infty} \|f_j\|_p = \|f\|_p$. Prove that f_j converges to f in L^p.

4. Prove the following particular case of the change of variable theorem for the Lebesgue integral:

If $f \in L^1(\mathbb{R}, m)$, then for any $a > 0$ and for any $b \in \mathbb{R}$,

$$\int_{\mathbb{R}} f(ax+b)dm(x) = \frac{1}{a}\int_{\mathbb{R}} f(x)dm(x).$$

5. Let $g \in L^1(\mathbb{R}, m)$ be nonnegative. Fix $p \in [1, \infty)$. For $f \in L^p(\mathbb{R}, m)$, let $T(f) = f * g$ (convolution of f and g). Prove that

$$\|T\|_{L^p \to L^p} = \|g\|_{L^1}.$$

Here

$$\|T\|_{L^p \to L^p} := \sup_{f : \|f\|_{L^p} = 1} \|T(f)\|_{L^p}.$$

6. Let $E \subset \mathbb{R}$ be Lebesgue measurable. Define

$$f(x) = \mathrm{dist}(x, E) = \inf\{|x - e| : e \in E\}.$$

Prove that for m a.e. $x \in E$, $\lim_{r\to 0+} \frac{f(x+r)}{r} = 0$.