Math 540 Comprehensive Examination

May 17, 2013

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Let $f \in L^p(\mathbb{R})$. Prove that for any $\varepsilon > 0$, there exists a measurable set E such that $m(E) < \infty$ and $\|f\|_p \leq \|f\chi_E\|_p + \varepsilon$.

2. Let $\{f_n\}$ be a sequence of complex-valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Determine whether the following statements are true. For the false statement, provide a counterexample. For the true one, prove it.
a) $\{f_n\}$ converges to f in L^1, then $f_n \to f$ in measure.
b) $f_n \to f$ a.e., then $f_n \to f$ in measure.
c) $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.

3. Let f be a measurable function on $(X, \mathbb{A}, \mu)$. Determine whether the following statements are true. For the false statement, provide a counterexample. For the true one, prove it.
a) if $f \in L^\infty$, then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.
b) if $f \in L^p$ for all $\infty \geq p \geq 1$, then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

4. Let E be a Lebesgue measurable subset of $\mathbb{R}$. Prove that

$$\lim_{x\to 0} m(E \cap (E + x)) = m(E).$$

Here $E + x = \{y + x : y \in E\}$.

5. Assume that $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing,

$$\int_{\mathbb{R}} f' = 1, \quad \lim_{x\to-\infty} f(x) = 0, \text{ and } \lim_{x\to\infty} f(x) = 1.$$

Prove that f is AC on any interval $[a, b]$.

6. Let f_n be a sequence of Lebesgue measurable functions on the interval $[0, 1]$. Assume that f_n converges to a function f m almost everywhere, and that

$$\int_{[0,1]} |f_n|^2 dm \leq 1$$

for each n. Prove that f_n converges to f in L^1.
Hint: Use Egoroff's thm.