Math 540 Exam

May, 2011

Calculators, books and notes are not allowed!

1. Prove by defintion that if $\mu_1, \cdots, \mu_n$ are measures on $(X, \mathcal{A})$, $a_1, \cdots, a_n \in [0, \infty)$, then $\sum_{j=1}^{n} a_j \mu_j$ is a measure on $(X, \mathcal{A})$.

2. Compute $\lim_{k\to\infty} \int_0^k x^n \left(1 - k^{-1}x\right)^k dx$. Here $n \in \mathbb{N}$.

3. Let μ^* be an outer measure on X. $\{A_j\}$ be a sequence of disjoint μ^*-measurable sets. Prove that $\mu^*\left(E \cap (\cup_j A_j)\right) = \sum_j \mu^*(E \cap A_j)$.

4. Suppose that $\{f_n\}$ is a sequence of positive measurable functions on $\mathcal{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$ at every $x \in \mathbb{R}$, and $\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n < \infty$. Prove that $\int_E f = \lim_{n\to\infty} \int_E f_n$ for all measurable sets E.

5. Let E be a Lebesgue measuarble set in $\mathbb{R}$ and $m(E) > 0$. Prove that for any $1 > \varepsilon > 0$, there is an open interval I such that $m(E \cap I) > \varepsilon m(I)$.

6. Let $f \in L^p(\mathbb{R})$ with $1 \le p < \infty$. Prove that

$$\lim_{\lambda\to 0} \lambda^p m(\{x \in \mathbb{R} : |f(x)| > \lambda\}) = \lim_{\lambda\to\infty} \lambda^p m(\{x \in \mathbb{R} : |f(x)| > \lambda\}) = 0$$