Math 540 Exam

May, 2010

Calculators, books and notes are not allowed!

1. Show that for $p > 1$,

$$\lim_{n\to\infty} \int_1^n \frac{\left(1-\frac{t}{n}\right)^n}{t^p} dm(t) = \int_1^\infty \frac{e^{-t}}{t^p} dm(t) .$$

Here m is the Lebesgue measure on $\mathbb{R}$.

2. Let

$$f(x) = \begin{cases} x\sin(1/x) & \text{for } 0 < x \le \infty \\ 0 & \text{for } x = 0 \end{cases}$$

(a) Is f is uniformly continuous on $[0,\infty)$? Prove your answer!
(b) Is f of bounded variation on $[0,\infty)$? Prove your answer!

3. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R})$. Prove that

$$\lim_{\delta\to 0} \int_{\mathbb{R}} |f(x+\delta) - f(x)|^p dx = 0 .$$

(*Hint: Use the fact that C_c^0 is dense in L^p. Here the space C_c^0 is the set consisting of all continuous functions with compact support.*)

4. (a) State Egoroff's theorem.
(b) State the Dominated Convergence Theorem.
(c) Prove the Dominated Convergence Theorem.

5. Let m be Lebesgue measure on $\mathbb{R}$. A sequence $\{f_n\}$ of measurable functions on $\mathbb{R}$ is said to *converge in measure* to the measurable function f if, given $\varepsilon > 0$, there exists an N such that

$$m\left(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon\}\right) < \varepsilon$$

for all $n \ge N$. Prove that
a) If $f_n \in L^p(\mathbb{R})$ and $\|f_n - f\|_p \to 0$ for some $1 \le p \le \infty$, then $f_n \to f$ in measure.
b) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to f a. e.

6. Let $\mathbb{Q}$ be the set of all rational numbers. A coset of $\mathbb{Q}$ in additive group $\mathbb{R}$ is a set $x + \mathbb{Q} = \{y \in \mathbb{R} : y = x + r \text{ for some } r \in \mathbb{Q}\}$. Let E be a set that contains exactly one point from each coset of $\mathbb{Q}$ in $\mathbb{R}$. Prove that
(a) $(r_1 + E) \cap (r_2 + E) = \emptyset$ if $r_1, r_2 \in \mathbb{Q}$ and $r_1 \ne r_2$
(b) $\mathbb{R} = \cup_{r\in\mathbb{Q}}(r + E)$.
(c) Prove that if $F \subset \mathbb{R}$ is a set such that every subset of F is Lebesgue measurable, then Lebesgue measure of F is 0.