Math 540 Comprehensive Examination

May 27, 2009

Solve problems (1), (2), (3) and (4). Also solve (only) **one** of problems (5) and (6).

Part I (solve all problems)

(1) (20 = 10 + 10 pts)

(a) Prove that if a sequence of functions (f_n) converges in measure to two functions f and g, then $f = g$ a.e.

(b) Give an example of a sequence (f_n) such that $f_n \to 0$ in measure on $[0, 1]$, yet $f_n(x) \not\to 0$ for each $x \in [0, 1]$.

(2) (20 = 10 + 10 pts) Let $1 \leq p \leq q \leq r \leq \infty$.

(a) Show that $L^p(\mathbb{R}) \cap L^r(\mathbb{R}) \subset L^q(\mathbb{R})$.

(b) For each p, q, r satisfying $1 \leq p < q < r \leq \infty$, give an example of a function f so that $f \in L^q(\mathbb{R})$, $f \notin L^p(\mathbb{R})$, and $f \notin L^r(\mathbb{R})$.

(3) (20 = 5 + 15 pts) Let H be a Hilbert space.

(a) Let W be a finite-dimensional subspace of H and let $\{x_1, \ldots, x_n\}$ be an orthonormal basis for W. For each $x \in H$ prove that there exist unique vectors $y, z \in H$ so that $x = y + z$, $y \in W$, and $\langle w, z\rangle = 0$ for all $w \in W$.

(b) Assume that H is infinite-dimensional. Let $\{x_n\}_{n=1}^{\infty}$ be a countably infinite orthonormal set of vectors in H. For each $x \in H$, show that the sequence (a_n) defined by $a_n := \langle x, x_n\rangle$ lies in ℓ^2, and that

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq ||x||^2.$$

(4) (20 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly increasing. Prove that the following are equivalent:

(i) f is absolutely continuous,

(ii) $f(\{x : f'(x) = +\infty\})$ has measure zero.

Part II (solve one of the following two problems)

(5) (20 = 10 + 10 pts) The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx.$$

(a) Show that the function $\hat{f}$ is bounded and continuous on $\mathbb{R}$.

(b) Show that $\lim_{|\xi| \to \infty} |\hat{f}(\xi)| = 0$.

(6) (20 pts) Let $f \in L^1([0,1])$. Assume that there is a constant c, $0 < c < 1$, so that the following holds: for every measurable set $A \subset [0,1]$ with $m(A) = c$, we have $\int_A f = 0$. Prove that $f = 0$ a.e.