Math 540 Comprehensive Examination

May 12, 2008

Solve problems (1),(2),(3) and (4). Also solve one of the problems (5),(6) or (7). In the following m stands for the Lebesgue measure on $\mathbb{R}$.

Good luck

Part I (Solve all problems)

1) Answer the following questions. Give short justifications for your answers.
   (a) (7 pts) State Fatou's Lemma and give an example for strict inequality.
   (b) (6 pts) Find a function of bounded variation which is not absolutely continuous.
   (c) (7 pts) Find a sequence $(f_n) \subset L^2[0,1]$ such that $\|f_n\|_2 = 1$ and

$$\lim_n \int_{[0,1]} f_n g \, dm \;=\; 0, \qquad \forall g \in L^2[0,1].$$

2) (a) (10 pts) Let $f \in L^4[0,1]$. Show that

$$\alpha_4(f) \;=\; \sup_{\lambda>0} \lambda^4 m(\{x : |f(x)| > \lambda\} \;\leq\; \|f\|_4^4.$$

   (b) (10 pts) Find an example of a measurable function f such that $\alpha_4(f)$ is finite, but f does not belong to $L^4[0,1]$.

3) (a) (15 pts) Let f be absolutely continuous on compact intervals and $f' \in L^p(\mathbb{R})$, $1 < p < \infty$. Show that

$$\sum_{n=-\infty}^{\infty} |f(n+1) - f(n)|^p \;< \infty.$$

   (b) (5 pts) Is (a) still true when f is continuous of bounded variation on every compact interval and $f' \in L^p(\mathbb{R})$?

4) (20 pts) Suppose $1 < p < \infty$ and $f \in L^p((0,\infty), m)$. Let

$$F(x) \;=\; \frac{1}{x}\int_0^x f(t)\, dm(t).$$

Show that

$$\|F\|_p \;\leq\; \frac{p}{p-1}\|f\|_p.$$

Hint: One possible way is to assume first that f is continuous with compact support. Integration by parts gives

$$\int_0^\infty F(x)^p \, dm(x) \;=\; -p\int_0^\infty F(x)^{p-1} x F'(x)\, dm(x).$$

What is $xF'(x)$?

Part II (Solve <u>one</u> of the following three)

5) (20 pts) Let f_n, f be functions in $L^1(\mathbb{R})$ and C be a constant such that $\|f_n - f\|_1 \leq \frac{C}{n^2}$ holds for all $n \geq 1$. Show that (f_n) converges to f almost everywhere.

6) (20 pts) Let $f : [a, b] \to [0, \infty)$ be measurable and $0 < \alpha \leq 1$. Show that

$$\lim_n \int_a^b n\left(\left(1 + \frac{f(x)}{n}\right)^\alpha - 1\right) dm(x)$$

exists and identify the limit.

7) (20 pts) Let $f \in L^\infty(\mathbb{R})$ be real-valued. Show that the essential range of f, defined as

$$R_f = \{y : \forall \varepsilon > 0,\ m(\{x : |f(x) - y| < \varepsilon\}) > 0\},$$

is a compact subset in $\mathbb{R}$.