Math 540 **Comprehensive Examination** **May 21, 2007**

Solve all four problems in Part I and **one** problem in Part II. Indicate your choice. Credit will be given only for one problem in Part II. Each problem is worth 20 points. m denotes the Lebesgue measure on $\mathbb{R}$.

Part I

I. Let $f \in L^1([0,1])$. For $t \geq 0$, let $F(t) = \int_0^1 xe^{-t/x} f(x)\, dm(x)$.

(i) (6 points) Prove that $F(t)$ is finite for all $t \geq 0$.
(ii) (7 points) Prove that $F : [0, \infty) \to \mathbb{R}$ is continuous.
(iii) (7 points) Is F differentiable? If so, calculate $F'(t)$ for $t > 0$ and $\lim_{h \to 0^+} \frac{F(h)-F(0)}{h}$.

II. Decide whether each of the following statements is true or false. Justify your answer with a short proof if the statement is true or a counterexample if it is false. (5 points each)

(a) Let (f_n) be a sequence in $L^p([0,1])$ which converges in L^p to $f \in L^p([0,1])$. Then (f_n) converges to f in measure. (A sequence (g_n) is said to *converge in measure* to g if $\lim_{n\to\infty} m(\{x : |g_n(x) - g(x)| \geq \epsilon\}) = 0$ for every $\epsilon > 0$.)
(b) If $f : [a, b] \to \mathbb{R}$ is absolutely continuous and one-to-one, then f^{-1} is absolutely continuous.
(c) If $f : [0,1] \to \mathbb{R}$ is a measurable function so that $\int_E f\, dm = 0$ for all measurable sets $E \subset [0,1]$, then $f = 0$ almost everywhere.
(d) If $f : [0,1] \to \mathbb{R}$ is continuous, then f is of bounded variation.

III. For $n \in \mathbb{N}$, let $h_n = \sum_{j=1}^n (-1)^j \chi_{[(j-1)/n, j/n]}$. Show that $\lim_{n\to\infty} \int_{[0,1]} f h_n\, dm = 0$ for every $f \in L^1([0,1])$. Here χ_E denotes the characteristic function of the set E.

IV. (i) (10 points) Let $f : [0,1] \to \mathbb{R}$ be a continuous function which is absolutely continuous on $[\epsilon, 1]$ for every $0 < \epsilon < 1$, and of bounded variation on $[0,1]$. Prove that f is absolutely continuous on $[0,1]$.
(ii) (10 points) Let $f : [0,1] \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq |x^{1/3} - y^{1/3}|$ for all $x, y \in [0,1]$. Must f be absolutely continuous? Justify your answer.

Part II

V. Let $f \in L^\infty([0,1])$. Prove the following statements:

(i) (10 points) The function $p \mapsto ||f||_p$ is nondecreasing for $1 \leq p < \infty$.
(ii) (10 points) $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

VI. Let $\mathcal{H}$ be the Hilbert space $L^2([0,1])$.

(i) (5 points) Prove the **Parallelogram Identity**:

$$||f+g||^2 + ||f-g||^2 = 2(||f||^2 + ||g||^2) \qquad \forall f, g \in \mathcal{H}.$$

(ii) (15 points) Let $K \subset \mathcal{H}$ be a nonempty, closed, convex set, and let $f \in \mathcal{H}$. Prove that there exists a unique element $h \in K$ so that $||f-h|| = \inf_{g\in K} ||f-g||$.