Math 540 Comprehensive Examination

May 16, 2006

Solve all five problems. All problems have equal value.

m denotes the Lebesgue measure on $\mathbb{R}$.

I. Prove the following equality by using an infinite series expansion. Justify the term-by-term integration.

$$\int_{[0,1]} \frac{\ln x}{x-1}\, dm(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Does this equality hold if the integral is regarded as a Riemann integral?

II. Suppose $f_n : [0,1] \to [0,\infty)$ are measurable, $\|f_n\|_2 \le 1$ for all n, and $f_n \to f$ a.e. on $[0,1]$.

(i) Show $f \in L^2[0,1]$.

(ii) Show $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Hint. One way of solving (ii) is by using Egoroff's theorem.

III. Suppose $f \in L^1(\mathbb{R}, m)$. Prove that for each $\varepsilon > 0$, there is $\delta > 0$ such that $\int_E |f|\, dm < \varepsilon$ whenever E is measurable and $m(E) < \delta$.

IV. (i) Let $f \in L^p([0,1], m)$, $1 < p < \infty$. Show

$$\lim_{y \to 0+} y^{\frac{1-p}{p}} \int_{[0,y]} f(x)\, dm(x) = 0.$$

(ii) Is it true that

$$\bigcap_{1 \le p < \infty} L^p(\mathbb{R}, m) \subseteq L^\infty(\mathbb{R}, m)\ ?$$

Justify your answer.

V. Consider the Banach space

$$\ell^p = \left\{ f : \mathbb{N} \to \mathbb{C} : \|f\|_p = \left(\sum_{n=1}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 < p < \infty.$$

Prove directly that for every element ϕ in the dual Banach space $(\ell^3)^*$ there is a unique element $g \in \ell^{\frac{3}{2}}$ such that

$$\phi(f) = \sum_{n=1}^{\infty} f(n) g(n).$$

What is the relation between $\|g\|_{\frac{3}{2}}$ and $\|\phi\|$?